

HARMONIC ANALYSIS ON THE INFINITE SYMMETRIC GROUP

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ABSTRACT. The infinite symmetric group $S(\infty)$, whose elements are finite permutations of $\{1, 2, 3, \dots\}$, is a model example of a “big” group. By virtue of an old result of Murray–von Neumann, the one–sided regular representation of $S(\infty)$ in the Hilbert space $\ell^2(S(\infty))$ generates a type II_1 von Neumann factor while the two–sided regular representation is irreducible. This shows that the conventional scheme of harmonic analysis is not applicable to $S(\infty)$: for the former representation, decomposition into irreducibles is highly non–unique, and for the latter representation, there is no need of any decomposition at all. We start with constructing a compactification $\mathfrak{S} \supset S(\infty)$, which we call the space of virtual permutations. Although \mathfrak{S} is no longer a group, it still admits a natural two–sided action of $S(\infty)$. Thus, \mathfrak{S} is a G –space, where G stands for the product of two copies of $S(\infty)$. On \mathfrak{S} , there exists a unique G –invariant probability measure μ_1 , which has to be viewed as a “true” Haar measure for $S(\infty)$. More generally, we include μ_1 into a family $\{\mu_t : t > 0\}$ of distinguished G –quasiinvariant probability measures on virtual permutations. By making use of these measures, we construct a family $\{T_z : z \in \mathbb{C}\}$ of unitary representations of G , called generalized regular representations (each representation T_z with $z \neq 0$ can be realized in the Hilbert space $L^2(\mathfrak{S}, \mu_t)$, where $t = |z|^2$). As $|z| \rightarrow \infty$, the generalized regular representations T_z approach, in a suitable sense, the “naive” two–sided regular representation of the group G in the space $\ell^2(S(\infty))$. In contrast with the latter representation, the generalized regular representations T_z are highly reducible and have a rich structure. We prove that any T_z admits a (unique) decomposition into a multiplicity free continuous integral of irreducible representations of G . For any two distinct (and not conjugate) complex numbers z_1, z_2 , the spectral types of the representations T_{z_1} and T_{z_2} are shown to be disjoint. In the case $z \in \mathbb{Z}$, a complete description of the spectral type is obtained. Further work on the case $z \in \mathbb{C} \setminus \mathbb{Z}$ reveals a remarkable link with stochastic point processes and random matrix theory.

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0. INTRODUCTION

0.1. The infinite symmetric group: characters, factor representations, spherical representations. The present paper deals with harmonic analysis on the infinite symmetric group $S(\infty)$ (the group of finite permutations of the infinite set $\{1, 2, \dots\}$). This group belongs to the class of *big groups*,¹ such as the infinite dimensional classical groups, diffeomorphism and current groups. We present here the first example of harmonic analysis on a big group. A short exposition of our results was published in [KOV]; here we provide the detailed statements and proofs.

The most well-studied groups in representation theory — compact, abelian, and reductive Lie groups — are *tame*. Irreducible unitary representations of tame groups can be classified, and the basic problem of harmonic analysis consists in decomposing interesting reducible representations (e.g., the regular representation) into irreducible components.

The group $S(\infty)$ is *wild*, not tame. The wild groups do not admit any sensible classification of irreducible representations, and reducible representations can be decomposed into irreducibles in essentially different ways. Unlike tame groups, the wild groups have factor representations of von Neumann types II and III. All those facts imply that the representation theory of wild groups should be based on the principles distinct from those used for tame groups.²

The possibility of developing such a theory was first demonstrated by E. Thoma [Tho1] exactly for the example of $S(\infty)$. In that paper Thoma discovered that the finite type factor representations of the group $S(\infty)$ admit an explicit classification. More precisely, those representations are labelled by the pairs $\omega = (\alpha; \beta)$ of nonincreasing sequences of nonnegative numbers

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq 0), \quad \beta = (\beta_1 \geq \beta_2 \geq \dots \geq 0)$$

subject to the extra condition

$$\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \beta_k \leq 1.$$

¹This term, introduced in Vershik [Ver], is somewhat vague but expressive and convenient. About representations of various big groups, see, e.g., Ismagilov [Ism], Neretin [Ner1], Olshanski [Ol4], Strătilă–Voiculescu [SV].

²For general facts about tame and wild groups, see, e.g., Dixmier [Dix], Kirillov [Kir]. Note that tame groups are also called *type I* groups. About factor representations, see, e.g., Dixmier [Dix], Thoma [Tho2].

The set of such pairs forms an infinite-dimensional simplex Ω which we call the *Thoma simplex*.³

For an arbitrary unitary representation of a countable group, generating a von Neumann algebra of finite type, a decomposition into a direct integral of factor representations always exists and is essentially unique. Therefore, the main problem of harmonic analysis on $S(\infty)$ may consist in the actual decomposition of most notable representations of finite type. This agrees with an old idea (especially developed by Pukanszky) that as “elementary representations” for a wild group, one should consider not irreducible representations but factor representations of an appropriate class.

Instead of decompositions of representations one can speak of decompositions of their *characters*. Denote by \mathcal{X} the set of positive definite functions χ on the group $S(\infty)$, constant on conjugacy classes and normalized at the identity permutation by the condition $\chi(e) = 1$. In the topology of pointwise convergence, the space \mathcal{X} is a convex compact set. It is known that the extreme points of the set \mathcal{X} are exactly the characters of factor representations of finite type. Moreover, the space \mathcal{X} is a Choquet simplex, that is, any point $\chi \in \mathcal{X}$ can be uniquely represented as a continual convex combination of extreme characters.

However, as was shown by one of the authors ([Ol2], [Ol3]), one can develop another approach to representation theory of the group $S(\infty)$, which makes it possible to avoid factor representations. Recall that, by definition, the group $S(\infty)$ consists of *finite* permutations of the set $\{1, 2, \dots\}$, fixing all but finitely many points. Consider now the group $\overline{S(\infty)}$ of *all* bijections of the set $\{1, 2, \dots\}$ onto itself. Taking the subsets

$$\overline{S_m(\infty)} = \{g \in \overline{S(\infty)} : g(k) = k, k = 1, \dots, m\},$$

where $m = 1, 2, \dots$, as a fundamental system of neighborhoods of identity we turn $\overline{S(\infty)}$ into a topological group.

The subgroup $S(\infty)$ is dense in $\overline{S(\infty)}$. Furthermore, denote by \overline{G} the group of pairs $(g_1, g_2) \in \overline{S(\infty)} \times \overline{S(\infty)}$, such that $g_1^{-1}g_2 \in S(\infty)$, and identify $\overline{S(\infty)}$ with the diagonal subgroup \overline{K} in \overline{G} . We introduce a topology in \overline{G} by proclaiming the subgroup \overline{K} open. The group $S(\infty) \times S(\infty)$ is dense in \overline{G} . The groups \overline{K} and \overline{G} are totally disconnected and *not* locally compact.

Remarkably enough, the topological group \overline{G} is tame and its irreducible representations can be completely described, see Olshanski [Ol3], Okounkov [Ok1], [Ok2]. In the present paper we are only interested in *spherical* irreducible representations of the group \overline{G} (that is, representations containing a \overline{K} -invariant vector). It is known that $(\overline{G}, \overline{K})$ is a *Gelfand pair*: in any irreducible spherical representation there is only one, up to a scalar factor, \overline{K} -invariant vector ξ .

There exists a natural bijection $T \leftrightarrow \pi$ between irreducible spherical representations T of the pair $(\overline{G}, \overline{K})$ and factor representations π of finite type of the group $S(\infty)$. Specifically, π is the restriction of T to the subgroup $S(\infty) \times \{e\} \subset \overline{G}$. The character $\chi(s) = \text{Tr } \pi(s)$ of the representation π (here “Tr” denotes the normalized trace on the factor) is related to the spherical function $\varphi(g) = (T(g)\xi, \xi)$ by the simple formula

$$\varphi(g_1, g_2) = \chi(g_1^{-1}g_2), \quad (g_1, g_2) \in \overline{G}.$$

³For more detail on Thoma’s theorem, see §9.6 below.

Therefore, we can think of *finite* factor representations of the group $S(\infty)$ as of irreducible spherical representations of the pair $(\overline{G}, \overline{K})$, thus returning to the conventional setup of representation theory of tame groups.

0.2. The generalized regular representations T_z and the problem of harmonic analysis. We shall now describe the representations of the group \overline{G} whose decomposition is the purpose of the present paper. The choice of those representations is itself a nontrivial problem. Traditionally, the object of harmonic analysis for a Gelfand pair $(\mathcal{G}, \mathcal{K})$ is the decomposition of the natural representation of \mathcal{G} in the space $L^2(\mathcal{G}/\mathcal{K})$ (for instance, for Riemannian symmetric spaces \mathcal{G}/\mathcal{K} , the decomposition problem was studied in classical works of Harish–Chandra and Gindikin–Karpelevich). In our situation the space $\overline{G}/\overline{K} \cong S(\infty)$ is discrete, so that the Hilbert space $L^2(\overline{G}/\overline{K}) = \ell^2(S(\infty))$ makes sense. However, the corresponding representation turns out to be irreducible so that there is no need of any further decomposition. This irreducibility effect is equivalent to the following fact (which is probably better known): the one-sided regular representation of the group $S(\infty)$ in the space $\ell^2(S(\infty))$ generates a type II_1 factor.⁴

Instead of the homogeneous space $\overline{G}/\overline{K} = S(\infty)$ we introduce a compact space \mathfrak{S} containing $S(\infty)$ as a dense subset. More precisely, the space \mathfrak{S} is defined as a projective limit of finite sets,

$$S(1) \leftarrow S(2) \leftarrow \dots \leftarrow S(n) \leftarrow \dots$$

Here $S(n)$ is the set of all permutations of n objects, and the projections are specified in §1 below. The points of \mathfrak{S} are called *virtual permutations*. Unlike $S(\infty)$, the space \mathfrak{S} is *not* a group. However, it admits a canonical action of the group \overline{G} , which is sufficient for our purposes. In particular, we have a two-sided action of the group $S(\infty)$ on \mathfrak{S} .

On the space $\mathfrak{S} \supset S(\infty)$ of virtual permutations there is a (unique) probability measure invariant with respect to the action of the group \overline{G} . This measure, which we denote as μ_1 , is much more interesting and useful than the counting (Haar) measure on $S(\infty)$. More generally, the measure μ_1 can be included into a one-parameter family of probability measures μ_t , $t > 0$. All those measures are invariant with respect to the subgroup \overline{K} , and quasiinvariant with respect to \overline{G} .⁵

By applying a standard general construction of producing unitary representations from a group action with quasiinvariant measure, we arrive at a family of representations T_z of the group \overline{G} . Here z ranges over the set $\mathbb{C} \setminus \{0\}$, and the representation T_z acts in the space $L^2(\mathfrak{S}, \mu_t)$ where $t = |z|^2$. One can also define two more unitary representations T_0, T_∞ in such a way that the resulting family is continuously parametrized by the points of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. The representation T_∞ coincides with the above mentioned two-sided regular representation in the space $\ell^2(S(\infty))$. We regard the family $\{T_z\}$ as a deformation of T_∞ ,

⁴More generally, for any discrete group whose conjugacy classes, except $\{e\}$, are infinite, the one-sided regular representation is a type II_1 factor representation, while the two-sided regular representation is irreducible. See Murray–von Neumann [MvN, chapter 5], Naimark [Nai, chapter VII, §38.5].

⁵The idea of extending the group space (of a big group) in order to obtain measures with good transformation properties comes from the measure theory on infinite-dimensional linear spaces and is rather old, see, e.g., Gelfand–Vilenkin [GV, chapter IV].

and we call T_z the *generalized regular representations*.⁶

The only irreducible representation in the family $\{T_z : z \in \mathbb{C} \cup \{\infty\}\}$ is T_∞ . All the representations T_z , $z \in \mathbb{C}$, are reducible, and their structure is rather complicated. *We consider the explicit decomposition of those representations as the main problem of harmonic analysis for the group $S(\infty)$.*

Each representation T_z can also be realized as an inductive limit of the two-sided regular representations of finite groups $S(n) \times S(n)$,

$$\text{Reg}^1 \rightarrow \text{Reg}^2 \rightarrow \dots \rightarrow \text{Reg}^n \rightarrow \dots, \quad (*)$$

with very special embeddings $\text{Reg}^n \rightarrow \text{Reg}^{n+1}$ depending on z . The existence of a finite dimensional approximation allows one to employ the powerful combinatorial and probabilistic techniques used in the theory of approximately finite-dimensional (AF-) algebras. In case of the group $S(\infty)$, these techniques, based on combinatorics of Young diagrams and the theory of symmetric functions, were developed in [VK2], [VK3].

0.3. Main results of the paper. These are as follows:

1. The representations T_z and $T_{\bar{z}}$ are equivalent, and we describe an intertwining operator realizing their equivalence.

2. For any $z \in \mathbb{C}$, the representation T_z can be decomposed into a multiplicity free direct integral of irreducible spherical representations. So, the equivalence class of T_z is completely described by an equivalence class of measures on the Thoma simplex Ω . We will refer to the latter equivalence class as to the *spectral type* of T_z .

3. The spectral types of two representations T_{z_1} and T_{z_2} are disjoint (mutually singular) whenever z_1 and z_2 are not equal or conjugate to each other. This means that there exist two disjoint Borel subsets in Ω supporting the measures from the spectral types of T_{z_1} and T_{z_2} , respectively. Equivalently, there is no intertwining operator between T_{z_1} and T_{z_2} .

4. The spectral type of T_z substantially depends on whether z is an integer or not. In the present paper we focus on the case when $z \in \mathbb{Z}$. Then we are able to describe the spectral type quite explicitly. Namely, let $\Omega(p, q)$ denote the subset of those pairs $(\alpha, \beta) \in \Omega$ for which $\alpha_1 + \dots + \alpha_p + \beta_1 + \dots + \beta_q = 1$ (consequently, all the remaining coordinates α_i and β_j vanish); notice that $\Omega(p, q)$ is a $(p+q-1)$ -dimensional face of the Thoma simplex Ω . Our result says that when $z = 0, \pm 1, \pm 2, \dots$, the spectral type of T_z is determined by the union of Lebesgue measures on the finite-dimensional faces $\Omega(p, q)$ with $p - q = z$ (which agrees with the result on disjointness of spectral types stated above). We also show that if $z \in \mathbb{C} \setminus \mathbb{Z}$, then the spectral type of T_z is concentrated “inside” the simplex Ω . That is, all faces $\Omega(p, q)$ are null sets with respect to the spectral type of T_z .

0.4. The case $z \in \mathbb{C} \setminus \mathbb{Z}$. This case is studied in detail in a series of papers by Borodin and Olshanski, see [Ol5], [Bor2], [BO3], the expository papers [BO1], [Ol6], and references therein. When $z \in \mathbb{C} \setminus \mathbb{Z}$, there exists a distinguished probability measure in the spectral type of T_z (in the present paper it is denoted as σ_z). The key idea is to assign to (Ω, σ_z) a *stochastic point process*, and to study its *correlation functions*. It turns out that the point processes thus obtained are close to those

⁶Notice a certain similarity between our family $\{T_z\}$ and Neretin’s deformation of the natural representation on $L^2(\mathcal{G}/\mathcal{K})$, where \mathcal{G}/\mathcal{K} is a Riemannian symmetric space, see Neretin [Ner3].

arising in random matrix theory. The link with random matrix theory seems to be especially interesting and promising.

The results of Borodin and Olshanski were obtained as a continuation of the project started in [KOV]. Together with the present work they provide a description of the spectral types for all representations T_z .

0.5. Organization of the paper.

In §1, we start with the definition of the *canonical projection* $p_n : S(n) \rightarrow S(n-1)$. Using the projections p_n we define the space $\mathfrak{S} = \varprojlim S(n)$ of virtual permutations. Then we describe four different concrete realizations of that space. In one of them, \mathfrak{S} turns into the product of an infinite sequence of finite sets. Finally, we show that \mathfrak{S} is a G -space, where $G = S(\infty) \times S(\infty)$, and we introduce an additive \mathbb{Z} -valued 1-cocycle for the action of G on \mathfrak{S} .

In §2, we study the family $\{\mu_t\}_{t>0}$ of probability measures on the space \mathfrak{S} . This is a deformation of the unique G -invariant probability measure (in our notation, μ_1). The measures μ_t are central, i.e., invariant with respect to the action of the diagonal subgroup $K \subset G$. We show that they turn into product measures in one of the realizations of \mathfrak{S} . Moreover, they are essentially the only central measures with this property. Applying Kakutani's theorem we show that the measures μ_t are pairwise disjoint. Then we show that any μ_t is quasiinvariant with respect to the action of the group G , i.e., for any $g \in G$, the shift μ_t^g of μ_t by g is a measure in the equivalence class of μ_t . We also calculate the Radon–Nikodym derivative μ_t^g/μ_t , which we will need later.

In §3, we construct the representations T_z in two different ways. First, we realize T_z , where $z \in \mathbb{C}^*$, in the Hilbert space $L^2(\mathfrak{S}, \mu_t)$, where $t = |z|^2$. Here we use a multiplicative 1-cocycle $\mathfrak{S} \times G \rightarrow \mathbb{C}^*$ which depends on z and is defined via the additive cocycle from §2. Second, we realize the same representation T_z as the inductive limit corresponding to a chain of embeddings (*), see above. Using the latter realization, it is easy to complete the family $\{T_z\}$ by two limit representations, T_0 and T_∞ . Finally, we present a transparent interpretation of the embeddings in (*).

In §4, we define and study a distinguished matrix coefficient of T_z . The representation T_z has a distinguished K -invariant vector ξ_0 : in the first realization, this is the identity function $f_0 \equiv 1$ on the space \mathfrak{S} . Though the space of K -invariant vectors is infinite dimensional, ξ_0 is the only K -invariant vector which is seen at once: constructing other examples of K -invariant vectors is already a nontrivial task (we do this in section 6). To the vector ξ_0 one associates a spherical function on G , $\varphi_z(g_1, g_2) = (T(g_1, g_2)\xi_0, \xi_0)$, and a character $\chi_z(s) = \varphi(s, e)$ of the group $S(\infty)$. We do not dispose of a simple expression for the values of the function χ_z on conjugacy classes of the group $S(\infty)$; instead of this, we find a very nice formula for the coefficients $M_z(\lambda)$ in the expansions

$$\chi_z|_{S(n)} = \sum_{\lambda \in \mathbb{Y}_n} M_z(\lambda) \tilde{\chi}^\lambda, \quad n = 1, 2, \dots, \quad (**)$$

where \mathbb{Y}_n denotes the set of Young diagrams with n boxes, and $\tilde{\chi}^\lambda$ is the normalized irreducible character of $S(n)$, indexed by λ . In other words, we get the “Fourier coefficients” of χ_z (see Theorem 4.2.1).

This explicit formula for $M_z(\lambda)$ plays a key role in the present paper.⁷ We derive from it the following results:

First, we prove that ξ_0 is a cyclic vector if (and only if) $z \in \mathbb{C} \setminus \mathbb{Z}$. This means that for nonintegral values of z , the spectral type of T_z is entirely determined by the decomposition of the matrix coefficient associated with ξ_0 .

Second, we prove the equivalence $T_z \sim T_{\bar{z}}$, which is not evident from the construction of the representations. We also exhibit an isometric operator intertwining T_z and $T_{\bar{z}}$.

Finally, following the general philosophy of [VK2], [VK3], we assign to the family $M_z = \{M_z^{(n)}\}$ the so-called transition probabilities $p_z(\lambda, \nu)$. Given $\lambda \in \mathbb{Y}_n$, the numbers $p_z(\lambda, \nu)$ form a probability distribution on the set of those diagrams ν that can be obtained from λ by adding a single box.

In §5, we deal with the realization of T_z as an inductive limit of regular representations of the finite groups $S(n) \times S(n)$. To any such inductive limit we assign a “transition function” defined on the edges of the Young graph. This leads to a description of the commutant of the representation. We explicitly calculate the transition function for the representations T_z . Using this, we get, for integral values of the parameter z , a decomposition of T_z into a direct sum of subrepresentations T_{pq} which we call the *blocks* of T_z .

In §6, we get a convenient realization of the subspace V_z of K -invariant vectors in the Hilbert space of T_z . In terms of this realization we construct, for any $z \in \mathbb{Z}$, a K -invariant vector in each block of T_z , and we calculate the spectral decomposition of the corresponding matrix coefficient.

In §7, we prove that all the vectors constructed in section 6 are cyclic vectors in the corresponding blocks. Together with the results of section 6 this provides us with a complete description of the spectral type of the representation T_z in case of integral z . The argument of this section turns out to be rather long. At the end we give an example illustrating the origin of one of the difficulties.

In §8, we deal with arbitrary values of the parameter z . Here we prove that the spectral types of the representations T_z are pairwise disjoint (except the equivalence $T_z \sim T_{\bar{z}}$).

In §9 (Appendix), we collected a number of necessary definitions and facts of general nature.

0.6. Concluding remarks. The measures μ_t involved in the construction of the representations T_z are very interesting objects in their own right. Each measure μ_t can be written as a projective limit $\varprojlim \mu_t^n$, as $n \rightarrow \infty$, where μ_t^n is a remarkable probability measure on the finite symmetric group $S(n)$. These measures μ_t^n were first discovered in the context of population genetics and were considered in many subsequent works (see, e.g., the encyclopedic article Tavaré–Ewens [TE] and references therein). We will call the measures μ_t the *Ewens measures*. The probability space (\mathfrak{S}, μ_t) is closely related to the *Chinese Restaurant Process* construction, see Aldous [Ald], Pitman [Pit, §3.1].

There is a similarity between the spectral decomposition of the characters χ_z attached to the representations T_z , and the decomposition of the measures μ_t into K -invariant ergodic components. In a certain sense, these two problems are dual to each other: the latter refers to the “group level” while the former refers to the

⁷For alternative derivations of the formula and generalizations, see Kerov [Ker3], Borodin’s appendix in Olshanski [Ol5], Borodin–Olshanski [BO2].

“group dual level”. Moreover, there exists a general scheme unifying both problems and providing an interpolation between them, see Kerov [Ker3], Kerov–Okounkov–Olshanski [KOO], Borodin–Olshanski [BO2]. The decomposition of the measures μ_t is governed by the *Poisson–Dirichlet distributions*, see Kingman [Kin1], [Kin2], and also Olshanski [Ol5].

There is a deep analogy between the infinite symmetric group $S(\infty)$ and the infinite-dimensional unitary group $U(\infty)$. This analogy becomes apparent when one compares the description of characters of both groups, given in the fundamental papers Thoma [Tho1] and Voiculescu [Voi]. The theory of harmonic analysis for $S(\infty)$, as developed in the present paper and the papers of Borodin and Olshanski mentioned above, also has a counterpart for the group $U(\infty)$, see Olshanski [Ol6] and Borodin–Olshanski [BO5].

In particular, the counterparts of the Ewens measures μ_t are the so-called *Hua–Pickrell measures*, see Borodin–Olshanski [BO4]. More generally, similar measures can be associated to all 10 infinite series of classical Riemannian symmetric spaces of compact type, see Neretin [Ner2]. A pioneer work in this direction is that of Pickrell [Pic]; our construction of the space \mathfrak{S} of virtual permutations was largely influenced by that paper.

As was first observed by Borodin, the expression $(**)$ for the characters χ_z can be analytically continued to provide a *complementary series* of characters. In the papers of Borodin and Olshanski, the characters of the complementary series are considered together with the characters χ_z , the latter being viewed as the *principal series* (for a justification of this terminology, see Okounkov [Ok3]; note that there also exists a *degenerate series*). A natural question is whether these series of characters (and the corresponding representations) exhaust all “reasonable” objects of harmonic analysis for $S(\infty)$. In this direction little is known. Using the idea of Borodin [Bor], Rozhkovskaya [Rozh] obtained an elegant combinatorial characterization of the characters χ_z and their analytic continuation. Kerov [Ker1] considered the so-called *Ewens–Pitman measures* generalizing the Ewens measures μ_t , as possible candidates for an extension of the basic construction of the representations T_z . It would be interesting to pursue further the study of this question.

§1. THE SPACE OF VIRTUAL PERMUTATIONS

1.1. Canonical projections. Let $S(n)$ be the group of permutations of the finite set $\{1, \dots, n\}$, the *symmetric group of degree n* . We identify $S(n)$ with the subgroup of permutations $s \in S(n+1)$ preserving the last element $n+1$, i.e., $s(n+1) = n+1$. The inductive limit of groups $S(n)$ with respect to these embeddings (i.e., the union of these groups) will be denoted as $S(\infty) = \varinjlim S(n)$. The elements of $S(\infty)$ are *finite* permutations of the set $\{1, 2, \dots\}$, fixing all but finitely many natural numbers. We call $S(\infty)$ the *infinite symmetric group*.

Given a permutation $\tilde{s} \in S(n+1)$, $n = 1, 2, \dots$, we define its *derivative permutation* (we borrow the term from dynamical systems theory, see [CFS, chapter 1, §5]) $s = \tilde{s}' \in S(n)$ as follows

$$s(i) = \begin{cases} \tilde{s}(i), & \text{if } \tilde{s}(i) \leq n, \\ \tilde{s}(n+1), & \text{if } \tilde{s}(i) = n+1, \end{cases}$$

where $i = 1, \dots, n$. The map $\tilde{s} \mapsto s$, denoted $p_{n,n+1}$, will be referred to as the *canonical projection* of $S(n+1)$ onto $S(n)$. Here is an alternative description of the

canonical projection in terms of the cycle structure of permutations. Depending on the position of the element $n + 1$ in the cycles of the permutation \tilde{s} we distinguish between two cases: $n + 1$ belongs to a cycle $(\dots \rightarrow i \rightarrow n + 1 \rightarrow j \rightarrow \dots)$ of length ≥ 2 , or $n + 1$ is a fixed point of \tilde{s} . In the former case we remove $n + 1$ out of its cycle, i.e., we replace this cycle with $(\dots \rightarrow i \rightarrow j \rightarrow \dots)$. In the latter case \tilde{s} already belongs to the subgroup $S(n) \subset S(n + 1)$, and we set $s = \tilde{s}$.

It is clear from the definition that the preimage of a permutation $s \in S(n)$ with respect to the canonical projection contains $n + 1$ permutations in $S(n + 1)$. In fact, in order to obtain a permutation $\tilde{s} \in p_{n,n+1}^{-1}(s)$ one should insert $n + 1$ in a cycle of s right before one of the elements $j = 1, \dots, n$, or take $n + 1$ as a new 1-cycle. Note that in the latter case $\tilde{s} = s$.

Yet another useful description of the canonical projection employs the following simple operation on graphs. It will be convenient to identify permutations with bipartite graphs. We associate with a permutation $s \in S(n)$ a graph with the vertex set $\{1, \dots, n; 1', \dots, n'\}$. Its edges are couples of the form (i, j') , where $s(i) = j$. The projection $p_{n,n+1} : \tilde{s} \mapsto s$ can now be described as follows. Take the graph of \tilde{s} , and add an extra edge connecting the vertices $(n + 1)$ and $(n + 1)'$. Then the graph of the derivative permutation s arises if one takes for the edges the paths connecting the vertices in $1, \dots, n$ with the vertices in $1', \dots, n'$.

Note that the group $S(n)$ acts by left and right multiplications on both $S(n)$ and $S(n + 1)$.

Proposition 1.1.1. *The canonical projection $p_{n,n+1}$ is equivariant with respect to two-sided action of the group $S(n)$. If $n \geq 4$, this is the only map $S(n + 1) \rightarrow S(n)$ with this property.*

Proof. The first claim is immediate from the description of the canonical projection in terms of bipartite graphs, and the obvious graphical interpretation of the product of permutations. Assume now that the a map $p : S(n + 1) \rightarrow S(n)$ is equivariant. Then, for each permutation $s \in S(n)$, we have $s^{-1}p(e)s = p(s^{-1}es) = p(e)$, where e is the identity permutation. Since e is the only central element in $S(n)$ for $n \geq 3$, this implies $p(e) = e$. By the same token, $s^{-1}p((n, n + 1))s = p((n, n + 1))$ for every permutation $s \in S(n - 1)$. If $n \geq 4$, then e is the only element of $S(n)$ commuting with all permutations in $S(n - 1)$ (for $n = 2, 3$ the transposition (12) also shares this property). Therefore, $n \geq 4$ implies $p((n, n + 1)) = e$. Since the group $S(n + 1)$ is made of just two double $S(n)$ -cosets (the group $S(n)$ itself and the class containing $(n, n + 1)$), it follows that $p = p_{n,n+1}$. \square

Remark 1.1.2. As it is clear from the proof, there are non-canonical projections $p : S(n + 1) \rightarrow S(n)$ for $n = 2, 3$. They are determined by the equalities $p((23)) = (12)$ and $p((34)) = (12)$ respectively. Note that there are lots of maps $S(n + 1) \rightarrow S(n)$ which are equivariant with respect to a one-sided (left or right) action of the group $S(n)$.

1.2. Virtual permutations. Consider the sequence

$$S(1) \leftarrow \dots \leftarrow S(n) \leftarrow S(n + 1) \leftarrow \dots$$

of canonical projections, and let

$$\mathfrak{S} = \varprojlim S(n)$$

denote the projective limit of the sets $S(n)$. By definition, the elements of \mathfrak{S} are arbitrary sequences $x = (x_n \in S(n))$, such that $p_{n,n+1}(x_{n+1}) = x_n$ for all $n = 1, 2, \dots$. The set \mathfrak{S} is a closed subset of the compact space of all sequences (x_n) , therefore it is a compact space itself. Let $p_n : \mathfrak{S} \rightarrow S(n)$ denote the natural projection, $n = 1, 2, \dots$. The group $S(\infty)$ can be identified with a subset of \mathfrak{S} via the map $s \mapsto x = (x_n)$, $x_n = s$ for sufficiently large n . In other words, $S(\infty) \subset \mathfrak{S}$ consists of the stable sequences (x_n) .

The subset $S(\infty)$ is dense in \mathfrak{S} , because $p_n(S(\infty)) = p_n(\mathfrak{S}) = S(n)$ for any n . Hence, the space \mathfrak{S} is a compactification of the discrete space $S(\infty)$. The elements of \mathfrak{S} will be called *virtual permutations* of the set $\{1, 2, \dots\}$.

The definition of the space \mathfrak{S} does not change if we remove from the projective limit any finite number of first terms, i.e., if we start from $S(n)$ instead of $S(1)$, where n is chosen arbitrarily. This simple observation will be tacitly used in what follows.

In particular, it implies that changing the canonical projections $p_{n,n+1}$ for $n = 2, 3$ by noncanonical ones (see Remark 1.1.2) does not affect the construction of the space \mathfrak{S} .

1.3. Realizations of the space of virtual permutations. We shall use four concrete realizations of the space \mathfrak{S} of virtual permutations:

- (1) as of the infinite product $\prod_{n=1}^{\infty} \{0, 1, \dots, n-1\}$;
- (2) as of the space of growing trees;
- (3) as of the space of decreasing maps of the set $\{0, 1, \dots\}$ in itself;
- (4) as of the space of cyclic structures on the set $\{1, 2, \dots\}$.

All the constructions use appropriate realizations of finite sets $S(n)$, and the canonical projections preserve the specific structures of those sets. In this sense, the above realizations are *natural*.

Proposition 1.3.1. *There exists a natural homeomorphism $x = (x_n) \mapsto i = (i_n)$ between the space \mathfrak{S} and the infinite product*

$$I = I_1 \times I_2 \times \dots \quad \text{where } I_n = \{0, 1, \dots, n-1\}.$$

Proof. Given an element $x = (x_n) \in \mathfrak{S}$, we define the sequence $i = (i_n) \in I$ as follows. Set $i_1 = 0$. For every $n = 1, 2, \dots$ the coordinate i_{n+1} encodes the relation of $s = x_n$ and $\tilde{s} = x_{n+1}$. Specifically, $i_{n+1} = 0$ means that $s = \tilde{s}$, and $i_{n+1} = j \in \{1, \dots, n\}$ means that the element $n+1$ is inserted in a cycle of s immediately before j . One can easily check that this correspondence is indeed a homeomorphism $\mathfrak{S} \rightarrow I$, where I is equipped with the product topology. \square

Likewise, we get a bijection $S(n) \rightarrow I_1 \times \dots \times I_n$. The vector $i(x) = (i_1, i_2, \dots, i_n)$ is called the *code* of a permutation $x \in S(n)$. We shall apply this terminology to virtual permutations, too.

We define a *finite increasing tree* as a rooted labelled tree with $n+1$ vertices, labelled by the numbers $0, 1, \dots, n$ in such a way that the root has label 0 and the labels increase along every path leading off the root (cf. [Sta], §1.3). *Countable increasing trees* are defined in a similar way.

Proposition 1.3.2. *There exists a natural bijection between virtual permutations and countable increasing trees.*

Proof. Let τ be a countable increasing tree. Removing vertices with the labels $n+1, n+2, \dots$ we obtain a finite increasing tree which we denote as τ_n . Thus, a tree τ can be considered as the union of a chain $\tau_1 \subset \tau_2 \subset \dots$ of finite increasing trees, where τ_i has $i+1$ vertices. Given a tree τ_n , there are $n+1$ options for the next tree τ_{n+1} , which are naturally indexed by the numbers $0, 1, \dots, n$. In fact, we can join the vertex $(n+1)$ to every one of the vertices $0, \dots, n$. Therefore, each countable increasing tree τ is determined by the sequence $i = (i_n) \in I$. One can easily check that the map $\tau \mapsto (i_n)$ provides a bijection onto I . From Proposition 1.3.1 we get a bijection between countable increasing trees, and virtual permutations. The above construction is quite similar to that of the bijection between permutations in $S(n)$, and increasing trees with $n+1$ vertices, cf. [Sta], §1.3. \square

Note that the bijection $\mathfrak{S} \rightarrow I$ identifies finite permutations $s \in S(\infty) \subset \mathfrak{S}$ with the sequences $i = (i_n)$ with only finitely many nonzero coordinates. There is also an obvious interpretation in terms of increasing trees.

The first two realizations of virtual permutations are actually very close to each other. The third realization is a version of the first one, too. We say that a map φ of the set $\{0, 1, \dots\}$ into itself is *decreasing* if $\varphi(0) = 0$ and $\varphi(n) < n$ for all $n > 0$. Setting $\varphi(n) = i_n$ for $n \geq 1$ we obtain a bijective correspondence between decreasing maps and the points of I .

In the next subsection we describe the forth realization of virtual permutations; its nature is rather different from the first three.

1.4. Cyclic structures. Let J be an arbitrary set. We define a *cyclic order* on J as a family of subsets $[i, j)$, called *arcs*, and labelled by ordered pairs i, j of distinct points in J . We assume that the following four axioms hold:

- (1) for each pair i, j (where $i \neq j$) the arcs $[i, j)$, $[j, i)$ do not intersect, and their union is J ;
- (2) for each arc $[i, j)$, $i \in [i, j)$ (and hence $j \notin [i, j)$);
- (3) if i, j, k are pairwise distinct, then one of the arcs $[i, j)$, $[i, k)$ is a strict subset of the other;
- (4) if the arc $[i, j)$ is a strict subset of $[i, k)$, then $[j, k) = [i, k) \setminus [i, j)$.

If $|J| = 1$, there is only one cyclic structure on J with the empty family of arcs.

Note that cyclic orders on a finite set J are in a bijective correspondence with cyclic permutations of the elements of J . In fact, given a cyclic permutation s of the set J , we define the corresponding cyclic order as follows: for each pair $i \neq j$ the arc $[i, j)$ consists of the points $i, s(i), \dots, s^{m-1}(i)$ where m is the least number with $s^m(i) = j$. On the contrary, for infinite sets there is no natural relation between cyclic orders and permutations.

More generally, we define a *cyclic structure* on a set J as a partition of J into nonempty disjoint subsets (called *cycles*) with a specified cyclic order on each cycle. Let $CS(J)$ denote the set of all cyclic structures on J . If J is finite, there is a natural bijection between $CS(J)$ and the set of all permutations of J . Specifically, we associate with a permutation s its cycle partition with the cyclic order on each cycle as defined above.

For every subset $J' \subset J$ there is a natural projection $CS(J) \rightarrow CS(J')$. Moreover, if J is a union of an increasing sequence of subsets (J_n) , then $CS(J)$ can be naturally identified with the projective limit of the sets $CS(J_n)$. This observation leads to the following result.

Proposition 1.4.1. *There exists a natural bijection between \mathfrak{S} and the set $CS(\{1, 2, \dots\})$ of cyclic structures on $\{1, 2, \dots\}$.*

Note that the bijection of Proposition 1.4.1 identifies $p_n : \mathfrak{S} \rightarrow S(n)$ with the projection $CS(\{1, 2, \dots\}) \rightarrow CS(\{1, \dots, n\}) = S(n)$.

We define the *cycles of a virtual permutation* $x \in \mathfrak{S}$ as the cycles of the corresponding cycle structure on the set $\{1, 2, \dots\}$.

In the second realization of \mathfrak{S} (see §1.3), the cycles of x correspond to those subtrees of the rooted tree τ associated with x which are one edge apart from the root of τ .

In the third realization of \mathfrak{S} , the cycles correspond to the orbits of the associated map φ of the set $\{1, 2, \dots\}$ to itself. Here, by definition, two numbers i, j belong to the same φ -orbit if there exist k, l , such that $\varphi^k(i) = \varphi^l(j)$.

1.5. The groups G and K , and their action on the space \mathfrak{S} . Set $G = S(\infty) \times S(\infty)$. We call G the infinite *bisymmetric* group. Let $K = \text{diag } S(\infty)$ denote the diagonal subgroup $\{(s, s) \in G : s \in S(\infty)\}$ in G . We shall also use parallel notation

$$G(n) = S(n) \times S(n), \quad K(n) = \text{diag } S(n) \subset G(n).$$

Clearly, the group K is isomorphic to $S(\infty)$. We consider $S(\infty)$ as a right homogeneous space $K \backslash G$, where the action of G is defined as follows

$$s \cdot g = g_2^{-1} s g_1, \quad s \in S(\infty), \quad g = (g_1, g_2) \in G = S(\infty) \times S(\infty).$$

Since the canonical projections $p_{n,n+1}$ are equivariant, this action can be naturally extended to the action $\mathfrak{S} \times G \rightarrow \mathfrak{S}$ defined as $x \cdot g = y$, where

$$y_n = x_n \cdot g \quad \text{for all } n \text{ large enough.}$$

Specifically, this equality holds whenever n is so large that $g \in G$ already lies in $G(n)$.

Note that G acts on \mathfrak{S} by homeomorphisms. Thus, this is the (only) continuous extension of the action of G on $K \backslash G$ to the space \mathfrak{S} .

Unfortunately, in all our realizations the description of this action of G is rather awkward. Only the action of the subgroup K can be easily described in terms of the forth realization of \mathfrak{S} . Indeed, the action of K on \mathfrak{S} is the only continuous extension of its action on $S(\infty)$ by conjugations. Under the identification of the space \mathfrak{S} with the space of cyclic structures on $\{1, 2, \dots\}$, this turns into the natural action of K (as a group of permutations of $\{1, 2, \dots\}$) on the set $CS(\{1, 2, \dots\})$.

1.6. The fundamental cocycle. Given $s \in S(n)$, denote by $[s]$ the number of cycles in s . It is important for what follows that the difference $[x \cdot g] - [x]$ can be defined correctly for all $x \in \mathfrak{S}$ and $g \in G$.

Proposition 1.6.1. (i) *There exists an integer valued function $c(x, g)$ on $\mathfrak{S} \times G$, uniquely defined by the following property: if n is large enough, so that $g \in S(n) \times S(n)$, then*

$$c(x, g) = [p_n(x \cdot g)] - [p_n(x)] = [p_n(x) \cdot g] - [p_n(x)].$$

(ii) *This function is an additive cocycle:*

$$c(x, g_1 g_2) = c(x, g_1) + c(x \cdot g_1, g_2).$$

(iii) *If $g \in K$, then $c(\cdot, g) \equiv 0$.*

Proof. (i) It suffices to prove the claim for g of the form (s, e) or (e, s) , where s is a transposition $(ij) \in S(\infty)$. In this case (i) can be easily checked directly. In fact, let $1 \leq i < j \leq n$. If i, j are both in the same cycle of $x \in S(n)$, then the multiplication by (ij) from the left or from the right splits this cycle into two; otherwise the two cycles of x containing the elements i and j merge into a single cycle of the product $x \cdot (ij)$ or $(ij) \cdot x$. On the other hand, if $x = p_{n, n+1}(\tilde{x})$, where $\tilde{x} \in S(n+1)$, then i, j belong to one and the same cycle of x if and only if they belong to one and the same cycle of \tilde{x} . Therefore, $[p_n(x) \cdot (ij)] - [p_n(x)] = [(ij) \cdot p_n(x)] - [p_n(x)] = \pm 1$, and this number does not change when n is replaced by $n+1$.

The claim (ii) follows from (i), and (iii) is obvious. \square

We call $c(x, g)$ the *fundamental cocycle* of the dynamical system (\mathfrak{S}, G) .

Remark 1.6.2. Let $\overline{G} \supset G$ and $\overline{K} \supset K$ be topological groups as defined above in §0.1. One can prove that the action of G on \mathfrak{S} can be extended to a continuous action $\mathfrak{S} \times \overline{G} \rightarrow \mathfrak{S}$. In particular, the subgroup $\overline{K} \subset \overline{G}$ also acts on \mathfrak{S} . Moreover, all claims of Proposition 1.6.1 hold when G and K are replaced by \overline{G} and \overline{K} , respectively.

2. QUASINVARIANT MEASURES

2.1. The G -invariant measure on \mathfrak{S} . Recall that $\mathfrak{S} = \varprojlim S(n)$ is the space of virtual permutations, and that we have defined an action of the bisymmetric group $G = S(\infty) \times S(\infty)$ on \mathfrak{S} . In what follows, all measures are Borel measures.

Proposition 2.1.1. *There exists a unique G -invariant probability measure μ_1 on \mathfrak{S} .*

Proof. Let μ_1^n denote the normalized Haar measure on $S(n)$. Clearly, μ_1^n coincides with the image of μ_1^{n+1} under the canonical projection $p_{n, n+1}: S(n+1) \rightarrow S(n)$. As $n \rightarrow \infty$, the projective limit measure

$$\mu_1 = \varprojlim \mu_1^n$$

on \mathfrak{S} is well defined. It is G -invariant, because the measures μ_1^k , $k \geq n$, are $G(n)$ -invariant for all $n = 1, 2, \dots$.

Conversely, let μ be a G -invariant probability measure on \mathfrak{S} . For any n , the push-forward of μ under the projection $p_n: \mathfrak{S} \rightarrow S(n)$ should coincide with μ_1^n , the only $G(n)$ -invariant probability measure on $S(n)$. Therefore, $\mu = \mu_1$. \square

Note that the assumption of G -invariance in Proposition 2.1.1 can be replaced by a weaker assumption of the invariance under the subgroups $S(\infty) \times \{e\} \subset G$ or $\{e\} \times S(\infty) \subset G$.

We think of μ_1 as of a substitute of Haar measure for the group $S(\infty)$.

2.2. Ewens measures μ_t . We shall now include the measure μ_1 into a one-parameter family of probability measures $\{\mu_t\}_{t>0}$ on \mathfrak{S} .

For $t > 0$ and arbitrary $n = 1, 2, \dots$ we define a measure μ_t^n on $S(n)$ by the formula

$$\mu_t^n(\{x\}) = \frac{t^{[x]}}{t(t+1) \dots (t+n-1)}, \quad x \in S(n).$$

Here, as in §1.6, $[x]$ stands for the number of cycles in x . The measure μ_t^n is a probability distribution, as it follows from a well-known identity (see Stanley [Sta, Proposition 1.3.4])

$$\sum_{k=1}^n c(n, k) t^k = t(t+1) \dots (t+n-1).$$

Here $c(n, k) = \#\{x \in S(n) : [x] = k\}$ is the absolute value of the Stirling number of the first kind. Another proof of this fact follows from

Proposition 2.2.1. *Under the bijection $S(n) \cong I_1 \times \dots \times I_n$ of §1.3, the measure μ_t^n turns into a product measure $\bar{\mu}_t^1 \times \bar{\mu}_t^2 \times \dots \times \bar{\mu}_t^n$. Here $\bar{\mu}_t^m$ is the measure on the set $I_m = \{0, 1, \dots, m-1\}$ defined as follows*

$$\bar{\mu}_t^m(\{k\}) = \begin{cases} \frac{1}{t+m-1}, & \text{if } k = 1, \dots, m-1, \\ \frac{t}{t+m-1}, & \text{if } k = 0. \end{cases}$$

Proof. Let $i(x) = (i_1, \dots, i_n) \in I_1 \times \dots \times I_n$ be the code of a permutation $x \in S(n)$, and let l denotes the number of 0's among the coordinates i_1, \dots, i_n . By the very definition,

$$(\bar{\mu}_t^1 \times \bar{\mu}_t^2 \times \dots \times \bar{\mu}_t^n)(i(x)) = \frac{t^l}{t(t+1) \dots (t+n-1)}.$$

On the other hand, a coordinate i_k vanishes if and only if the element k creates a new cycle of the permutation $p_k(x)$, hence the number of cycles $[x]$ in x coincides with the number l of zeros in the vector $i(x)$. This concludes the proof. \square

Corollary 2.2.2. *Given $t > 0$, the canonical projections $p_{n-1,n}$ preserve the measures μ_t^n , hence the measure*

$$\mu_t = \varprojlim \mu_t^n$$

on \mathfrak{S} is correctly defined. Under the identification $\mathfrak{S} \cong I$ of §1.3, the measure μ_t looks as the product measure

$$\bar{\mu}_t = \bar{\mu}_t^1 \times \bar{\mu}_t^2 \times \dots$$

Proof. Indeed, the canonical projection $p_{n-1,n}$ corresponds to deleting the last entry i_n of the code $(i_1, i_2, \dots, i_n) \in I_1 \times \dots \times I_n$. This immediately implies the both claims. \square

The measures μ_t^n on the groups $S(n)$ are known as *Ewens measures* (see §0.6). We will use the same name for the measures μ_t on the space \mathfrak{S} , which are built from the measures μ_t^n .

Proposition 2.2.3. *For any $t > 0$, the Ewens measure μ_t is invariant under the action of the group K on \mathfrak{S} .*

Proof. Indeed, it suffices to prove that for any n , the measure μ_t^n on $S(n)$ is invariant with respect to the action of the subgroup $K(n) \subset K$, isomorphic to $S(n)$. The action under question is simply the action of $S(n)$ on itself by conjugations. Since the measure μ_t^n on $S(n)$ has constant weights on conjugacy classes, it is invariant. \square

Since the measures μ_t , $0 < t < \infty$, live on a compact space, it is natural to ask for their limits as t goes to 0 or to $+\infty$.

Proposition 2.2.4. *There exist weak limits*

$$\text{w-lim}_{t \rightarrow 0} \mu_t = \mu_0, \quad \text{w-lim}_{t \rightarrow +\infty} \mu_t = \mu_\infty.$$

Here μ_0 is supported by the subset of virtual permutations with a single cycle while μ_∞ is the Dirac measure at the point $e \in S(\infty) \subset \mathfrak{S}$. Under the identification $\mathfrak{S} \cong I$, both μ_0 and μ_∞ become product measures.

Proof. Let us deal with the realization $\mathfrak{S} \cong I$. Then it suffices to examine the limit behavior of the measure $\bar{\mu}_t^n$ on $I_n = \{0, \dots, n-1\}$, where n is fixed and t goes to 0 or ∞ .

When $t \rightarrow 0$, the limit exists and is the measure $\bar{\mu}_0^n$ such that

$$\bar{\mu}_0^n(0) = 0, \quad \bar{\mu}_0^n(1) = \dots = \bar{\mu}_0^n(n-1) = \frac{1}{n-1}.$$

This means that the finite product measure $\bar{\mu}_0^1 \times \dots \times \bar{\mu}_0^n$, being transferred to $S(n)$, lives on maximal cycles in $S(n)$. Therefore, the infinite product measure $\bar{\mu}_0^1 \times \bar{\mu}_0^2 \times \dots$ on I , being transferred to \mathfrak{S} , lives on the virtual permutations with a single cycle in the sense of §1.4.

When $t \rightarrow \infty$, the measure $\bar{\mu}_t^n$ tends to the Dirac measure at $0 \in I_n$, which we denote as $\bar{\mu}_\infty^n$. This means that the measure on \mathfrak{S} corresponding to the infinite product $\bar{\mu}_\infty^1 \times \bar{\mu}_\infty^2 \times \dots$ is simply the Dirac measure at the point x such that $i(x) = (0, 0, \dots)$. This point is just e . \square

2.3. K -invariant product measures on I .

We can characterize the family of measures μ_t as follows.

Proposition 2.3.1. *The measures μ_t , $0 \leq t \leq \infty$, are precisely those probability measures on \mathfrak{S} that are both product measures (with respect to the identification $\mathfrak{S} \cong I$ of subsection 1.3) and invariant under K .*

Proof. Let μ be a K -invariant product measure on \mathfrak{S} . Clearly, $\mu = \varprojlim \mu^n$ with $\mu^n = p_n(\mu)$. We have to show that μ coincides with one of the measures μ_t , $0 \leq t \leq \infty$.

Consider the measure μ^2 on $S(2)$. It coincides with some μ_t^2 , $0 \leq t \leq \infty$, and the parameter t is determined uniquely. In fact, μ_∞^2 is the Dirac measure at $e \in S(2)$, μ_0^2 is the Dirac measure at the involution $(1, 2) \in S(2)$, and the measure μ_t^2 with $0 < t < \infty$ has weights $\frac{1}{1+t}$ and $\frac{t}{1+t}$ at the elements $(1, 2)$ and $e = (1)(2)$ respectively. We shall prove by induction in $n \geq 2$ that $\mu^n = \mu_t^n$ for all n .

We start by considering the induction step $n \rightarrow n+1$ in the degenerated cases of $t = 0$ and $t = \infty$. Here we need not the assumption that μ is a product measure. In the first case we assume that μ^n is the uniform distribution on n -cycles in $S(n)$. The preimage of n -cycles in $S(n+1)$ under the canonical projection is the set $A \cup B \subset S(n+1)$ where A consists of all n -cycles in $S(n) \subset S(n+1)$, and B is the set of all $(n+1)$ -cycles. Each of the two sets is a $\text{diag } S(n)$ -orbit, and B (but not A) is also a $\text{diag } S(n+1)$ -orbit. Since μ^{n+1} is $\text{diag } S(n+1)$ -invariant, it is supported by B alone and uniform on B .

The case $t = \infty$ is similar. We assume that μ^n is supported by the point $\{e\}$. Then the measure μ^{n+1} is supported by $A \cup B \subset S(n+1)$ where A consists of transpositions $(n+1, j)$, $1 \leq j \leq n$, and $B = \{e\}$. Since A is not $S(n+1)$ -invariant, μ^{n+1} is supported by B .

Let us now assume that $\mu^n = \mu_t^n$ where $t \neq 0, \infty$. We have to show that $\mu^{n+1} = \mu_t^{n+1}$. We shall write an arbitrary permutation $x \in S(n+1)$ as a pair $\{y, j\} \in S(n) \times I_{n+1}$ where $y = p_{n,n+1}(x)$ and

$$j = \begin{cases} 0 & \text{if } x(n+1) = n+1; \\ x(n+1) & \text{if } x(n+1) \neq n+1. \end{cases}$$

Since μ^{n+1} is a product measure,

$$\mu^{n+1}(\{x\}) = \frac{t^{[y]} \nu(j)}{t(t+1) \dots (t+n-1)}, \quad (2-3-1)$$

where $\nu(0), \dots, \nu(n)$ are the weights of a probability measure on I_{n+1} .

Applying a conjugation by appropriate permutation in $S(n) \subset S(n+1)$, one can replace any $j \neq 0$ with any other $j' \neq 0$ leaving the cycle structure of y intact. It follows that $\nu(1) = \dots = \nu(n)$, and we only have to find out the relation of $\nu(0)$ and $\nu(1) = \dots = \nu(n)$. To this end, choose a permutation $y \in S(n)$ such that the cycle containing 1 is of length ≥ 2 , i.e., $y(1) = j \neq 1$. Then

$$(1, n+1) \cdot \{y, 0\} \cdot (1, n+1) = \{y', j\}.$$

Here y' is obtained from y by removing j from its cycle and forming an additional trivial cycle (j) . An important point is that the initial cycle containing j does not disappear completely. It follows that $[y'] = [y] + 1$. Since, by the invariance assumption, $\mu^{n+1}(\{y, 0\}) = \mu^{n+1}(\{y', j\})$, we obtain from (2-3-1)

$$\nu(0) = t \nu(j) = t \nu(1).$$

Since

$$\nu(0) + n \nu(1) = 1,$$

we obtain

$$\nu(0) = \frac{t}{t+n}, \quad \nu(1) = \frac{1}{t+n},$$

and the desired identity $\mu^{n+1} = \mu_t^{n+1}$ follows. \square

2.4. Disjointness of the measures μ_t .

Proposition 2.4.1. *The measures μ_t , $0 \leq t \leq \infty$, are disjoint (mutually singular).*

Proof. We can replace the measures μ_t on \mathfrak{S} with the corresponding product measures

$$\bar{\mu}_t = \prod_{n=1}^{\infty} \bar{\mu}_t^n$$

on $I = I_1 \times I_2 \times \dots$. Note that $\bar{\mu}_\infty$ is the Dirac measure at the point $i = (0, 0, \dots) \in I$, and $\bar{\mu}_0$ is supported by the sequences $i = (i_1, i_2, \dots)$ with nonzero coordinates: $i_n \neq 0$ for $n \geq 2$. Obviously, the measures $\bar{\mu}_0$, $\bar{\mu}_\infty$ and $\bar{\mu}_t$ are pairwise disjoint, for every $t \in (0, \infty)$.

Assume now that $s, t \in (0, \infty)$ and $s \neq t$. We shall show that μ_s and μ_t are disjoint, $\mu_s \perp \mu_t$, by applying the well-known Kakutani criterion [Kak]. To this end we check that the infinite product $\prod_{n=1}^{\infty} a_n$, where

$$a_n = \sum_{i=0}^{n-1} \sqrt{\bar{\mu}_s^n(i) \bar{\mu}_t^n(i)},$$

diverges. Set $u = \sqrt{st}$; then

$$\begin{aligned} a_n &= \frac{u + n - 1}{\sqrt{(s + n - 1)(t + n - 1)}} = \\ &= \frac{1 + \frac{u}{n-1}}{\sqrt{(1 + \frac{s}{n-1})(1 + \frac{t}{n-1})}} = \\ &= 1 + \frac{u - \frac{1}{2}(s+t)}{n-1} + O\left(\frac{1}{(n-1)^2}\right). \end{aligned}$$

Now note that

$$u - \frac{1}{2}(s+t) = \sqrt{st} - \frac{1}{2}(s+t) \neq 0,$$

since $s \neq t$. Therefore, the product $\prod_{n=1}^{\infty} a_n$ is indeed divergent. \square

2.5. Quasiinvariance.

Proposition 2.5.1. *Each of the measures μ_t , $0 < t < \infty$ is quasiinvariant with respect to the action of G on the space \mathfrak{S} . More precisely,*

$$\frac{\mu_t(dx \cdot g)}{\mu_t(dx)} = t^{c(x,g)}; \quad x \in \mathfrak{S}, g \in G \quad (2-5-1)$$

where $c(x, g)$ is the fundamental additive cocycle of subsection 1.6. The measures μ_0 and μ_∞ are not quasiinvariant.

Proof. It suffices to check that

$$\mu_t(V \cdot g) = \int_V t^{c(x,g)} \mu_t(dx), \quad g \in G \quad (2-5-2)$$

for every Borel subset $V \subseteq \mathfrak{S}$. This would imply (2-5-1), hence the quasiinvariance of μ_t .

Fix $g \in G$ and choose m so large that $g \in G(m)$. For arbitrary $n \geq m$ and $y \in S(n)$, let $V_n(y) \subset \mathfrak{S}$ denote the preimage of the point $y \in S(n)$ under the canonical projection $p_n: \mathfrak{S} \rightarrow S(n)$; this is a cylinder set. It suffices to check (2-5-2) for $V = V_n(y)$.

Note that $V_n(y) \cdot g = V_n(y \cdot g)$ and $\mu_t(V_n(y)) = \mu_t^n(\{y\})$, hence

$$\mu_t(V_n(y) \cdot g) = \mu_t^n(\{y \cdot g\}).$$

On the other hand,

$$c(x, g) = [p_n(x \cdot g)] - [p_n(x)] = [y \cdot g] - [y]; \quad x \in V(y),$$

so that

$$t^{c(x, y)} \equiv t^{[y \cdot g] - [y]}; \quad x \in V_n(y).$$

The equation (2-5-2) takes the form

$$\mu_t^n(\{y \cdot g\}) = t^{[y \cdot g] - [y]} \mu_t^n(\{y\}),$$

which is immediate from the definition of the measure μ_t^n . \square

Using a well-known trick from ergodic theory, one can replace quasiinvariant measures μ_t by invariant, though infinite, measures. In order to do this, consider the space $\tilde{\mathfrak{S}} = \mathfrak{S} \times \mathbb{Z}$ and define an action of G on $\tilde{\mathfrak{S}}$ as

$$(x, k) \cdot g = (x \cdot g, k + c(x, g)), \quad x \in \mathfrak{S}, \quad k \in \mathbb{Z}, \quad g \in G. \quad (2-5-3)$$

Define the infinite measure ν_t on \mathbb{Z} as

$$\nu_t(\{k\}) = t^{-k}, \quad k \in \mathbb{Z},$$

and introduce the infinite measure $\tilde{\mu}_t = \mu_t \times \nu_t$ on $\tilde{\mathfrak{S}}$.

Proposition 2.5.2. *For every $0 < t < \infty$, the measure $\tilde{\mu}_t$ on the space $\tilde{\mathfrak{S}}$ is invariant with respect to the action (2-5-3) of the group G .*

Proof. This is immediate from (2-5-1) and the definition of the measure ν_t . \square

This construction will be used below, see §3.1.

Remark 2.5.3. One can prove that for every $t > 0$, the action of each of the subgroups $S(\infty) \times \{e\}$ and $\{e\} \times S(\infty)$ on the space \mathfrak{S} with the measure μ_t is ergodic and topologically minimal (the latter means that every orbit is dense in \mathfrak{S}). These claims will not be used in the sequel.

3. GENERALIZED REGULAR REPRESENTATIONS

In this Section we introduce a family $\{T_z\}$ of unitary representations of the group $G = S(\infty) \times S(\infty)$ parameterized by points $z \in \mathbb{C} \cup \{\infty\}$ of the Riemann sphere. First we assume $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

3.1. The representations T_z . We shall always assume below that the parameters $t > 0$ and $z \in \mathbb{C}^*$ are related as $t = z\bar{z}$. By virtue of Proposition 2-5-1,

$$\frac{\mu_t(dx \cdot g)}{\mu_t(dx)} = t^{c(x,g)} = |z^{c(x,g)}|^2, \quad x \in \mathfrak{S}, \quad g \in G.$$

Recall that $c(x, g)$ is an additive cocycle, so that $z^{c(x,g)}$ is a multiplicative one. Therefore, the following formula allows one to define a unitary representation T_z of the group G in the Hilbert space $\mathcal{H} = L^2(\mathfrak{S}, \mu_t)$,

$$(T_z(g)f)(x) = f(x \cdot g) z^{c(x,g)}, \quad g \in G, \quad x \in \mathfrak{S}, \quad f \in \mathcal{H}.$$

Note two cases when the multiplier $z^{c(x,g)}$ is equal to 1:

- (1) if $z = 1$ (then the measure $\mu_t = \mu_1$ is invariant);
- (2) if z is arbitrary but $g \in K \subset G$ (since $c(x, \cdot) \equiv 0$ on the subgroup K).

By the reasons to be made clear later on we call representations T_z the *generalized regular* representations of the group G .

Remark 3.1.1. The above construction is nothing but a specialization of a well-known general construction. Indeed, to any triple $(\mathfrak{X}, \mathcal{G}, \mu)$, where \mathcal{G} is a group acting on the right on a space \mathfrak{X} with a quasiinvariant measure μ , one associates a one-parameter family of unitary representations acting in $L^2(\mathfrak{X}, \mu)$ according to the formula

$$(T(g)f)(x) = f(x \cdot g) \left(\frac{\mu(dx \cdot g)}{\mu(dx)} \right)^{\frac{1+is}{2}}, \quad g \in \mathcal{G}, \quad x \in \mathfrak{X}, \quad f \in L^2(\mathfrak{X}, \mu),$$

where $s \in \mathbb{R}$ is a parameter. When $(\mathfrak{X}, \mathcal{G}, \mu) = (\mathfrak{S}, G, \mu_t)$, we obtain

$$\left(\frac{\mu(dx \cdot g)}{\mu(dx)} \right)^{\frac{1+is}{2}} = \left(t^{c(x,g)} \right)^{\frac{1+is}{2}} = z^{c(x,g)},$$

where $z = t^{\frac{1+is}{2}}$. Hence, we return to the definition of T_z .

One can introduce representations T_z in a slightly different way, using the action

$$g : (x, k) \longmapsto (x, k) \cdot g = (x \cdot g, k + c(x, g))$$

of the group G on the space $\tilde{\mathfrak{S}} = \mathfrak{S} \times \mathbb{Z}$ with the infinite invariant measure $\tilde{\mu}_t = \mu_t \times \nu_t$, see Proposition 2.5.2. There is a natural unitary representation \tilde{T}_t of the group G in the Hilbert space $L^2(\tilde{\mathfrak{S}}, \tilde{\mu}_t)$,

$$(\tilde{T}_t(g)f)(x, k) = f((x, k) \cdot g), \quad g \in G.$$

Let \mathbb{T} denote the unit circle $\{\xi \in \mathbb{C} : |\xi| = 1\}$.

Proposition 3.1.2. *For any $t > 0$, the representation \tilde{T}_t of the group G is unitary equivalent to the direct integral*

$$\int_{\xi \in \mathbb{T}} T_{\xi\sqrt{t}} d\theta, \quad \xi = e^{2\pi i\theta},$$

of the generalized regular representations T_z with $|z|^2 = t$.

Proof. Define a unitary representation of the group \mathbb{Z} acting in the space of \tilde{T}_t and commuting with the latter representation:

$$(\tilde{T}(l)f)(x, k) = f(x, k + l) t^{-l/2}, \quad l \in \mathbb{Z}.$$

We claim that the decomposition in question is determined by this commuting representation of \mathbb{Z} . To see this, we shall pass to a slightly different realization of \tilde{T}_t .

Consider the Hilbert space $L^2(\mathfrak{S} \times \mathbb{T}, \mu_t \times d\theta)$, where $d\theta$ is the normalized Lebesgue measure on the circle \mathbb{T} (we again write $\xi = e^{2\pi i\theta}$). In this space, we introduce two commuting unitary representations of the groups G and \mathbb{Z} , as follows

$$\begin{aligned} (\hat{T}_t(g)\hat{f})(x, \xi) &= \hat{f}(x \cdot g, \xi) (\xi\sqrt{t})^{c(x, g)}, \quad g \in G, \\ (\hat{T}(l)\hat{f})(x, \xi) &= \hat{f}(x, \xi) \xi^l, \quad l \in \mathbb{Z}, \end{aligned}$$

where \hat{f} ranges over $L^2(\mathfrak{S} \times \mathbb{T}, \mu_t \times d\theta)$.

Clearly, the representation \hat{T}_t admits the required decomposition, which is determined by the action of \mathbb{Z} . Hence, it suffices to check that the representation $\tilde{T}_t \times \tilde{T}$ in the space $L^2(\mathfrak{S}, \tilde{\mu}_t)$ is unitary equivalent to the representation $\hat{T}_t \times \hat{T}$ of the group $G \times \mathbb{Z}$ in the space $L^2(\mathfrak{S} \times \mathbb{T}, \mu_t \times d\theta)$.

The desired unitary equivalence is provided by the transform $\mathcal{F}: f \mapsto \hat{f}$,

$$\hat{f}(x, \xi) = \sum_{k=-\infty}^{\infty} f(x, k) t^{-k/2} \xi^{-k}, \quad (3-1-1)$$

which is, in essence, the Fourier transform with respect to the second argument.

Clearly, \mathcal{F} is an isometry. Let us check that it intertwines both representations of G :

$$\begin{aligned} \mathcal{F}(\tilde{T}_t(g)f)(x, \xi) &= \sum_{k=-\infty}^{\infty} (\tilde{T}_t(g)f)(x, k) t^{-\frac{k}{2}} \xi^{-k} = \\ &= \sum_{k=-\infty}^{\infty} f(x \cdot g, k + c(x, g)) (\xi\sqrt{t})^{-k} = \\ &= \sum_{j=-\infty}^{\infty} f(x \cdot g, j) (\xi\sqrt{t})^{-j+c(x, g)} = \\ &= \hat{T}_t(\mathcal{F}(f))(x, \xi). \end{aligned}$$

The intertwining property for the action of the group \mathbb{Z} can be checked in a similar way.

□

3.2. Admissibility. The definition of admissible representations is given in §9.9.

Proposition 3.2.1. *All the representations T_z , $z \in \mathbb{C}^*$, are admissible representations of the pair (G, K) .*

Proof. Given $n = 1, 2, \dots$, consider the canonical projection $p_n: \mathfrak{S} \rightarrow S(n)$. A function $F \circ p_n$, where F is any function on $S(n)$, will be called a *cylinder function* of level n on the space \mathfrak{S} . We denote the space of such functions by $\text{Cyl}^n(\mathfrak{S})$, and we call

$$\text{Cyl}(\mathfrak{S}) = \bigcup_{n \geq 1} \text{Cyl}^n(\mathfrak{S})$$

the *space of cylinder functions* on \mathfrak{S} . Clearly,

$$\text{Cyl}(\mathfrak{S}) \subset \mathcal{H} = L^2(\mathfrak{S}, \mu_t)$$

for every $t > 0$.

Note that the canonical projection $S(n) \rightarrow S(m)$ is invariant with respect to conjugations with the elements of the subgroup $S_m(n) \subset S(n)$, for all $m < n$. It follows that

$$p_m(x \cdot u) = p_m(x), \quad x \in \mathfrak{S}, \quad u \in K_m.$$

Since the factor $z^{c(x,g)}$ is trivial on the group $K \subset G$, we derive that

$$\text{Cyl}^m(\mathfrak{S}) \subseteq \mathcal{H}_m.$$

The space $\text{Cyl}(\mathfrak{S})$ is clearly dense in $\mathcal{H} = L^2(\mathfrak{S}, \mu_t)$, hence \mathcal{H}_∞ is also dense in \mathcal{H} , and the representation T_z is admissible. \square

Remark 3.2.2. The space $\text{Cyl}^m(\mathfrak{S})$ is strictly smaller than \mathcal{H}_m . Indeed, the former space is finite-dimensional for any m , while the latter space (as we shall see later on) has infinite dimension even for $m = 0$.

Remark 3.2.3. According to a general result (see §9.9), Proposition 3.2.1 implies that the representations T_z can be continued to the topological group \overline{G} . This can be verified directly by making use of Remark 1.6.2.

3.3. Approximations by regular representations. For every $n = 1, 2, \dots$ we denote by H^n the finite dimensional space $L^2(S(n))$ defined by the normalized Haar measure μ^n on $S(n)$, and by Reg^n the two-sided regular representation of the group $G(n) = S(n) \times S(n)$ in this space:

$$(\text{Reg}^n(g)f)(x) = f(g_2^{-1}xg_1)$$

where

$$g = (g_1, g_2) \in G(n), \quad x \in S(n), \quad f \in H^n.$$

We shall show that every generalized regular representation T_z can be obtained as an inductive limit of the representations Reg^n determined by an appropriate family of isometric embeddings $L_z^n: H^n \rightarrow H^{n+1}$, $n = 1, 2, \dots$, depending on $z \in \mathbb{C}^*$.

We define the operators $L_z^n: H^n \rightarrow H^{n+1}$ as follows: if $f \in H^n$ and $x \in S(n+1)$,

$$(L_z^n f)(x) = \begin{cases} z \sqrt{\frac{n+1}{t+n}} f(x) & \text{if } x \in S(n) \subset S(n+1); \\ \sqrt{\frac{n+1}{t+n}} f(p_n(x)) & \text{if } x \in S(n+1) \setminus S(n). \end{cases} \quad (3-3-1)$$

Here and below we assume that $t = z\bar{z}$.

Proposition 3.3.1. *For any $z \in \mathbb{C}^*$ the operator L_z^n provides an isometric embedding $H^n \rightarrow H^{n+1}$ which intertwines the $G(n)$ -representations Reg^n and $\text{Reg}^{n+1} \upharpoonright_{G(n)}$. Let T'_z denote the inductive limit of the representations Reg^n with respect to the embeddings*

$$H^1 \xrightarrow{L_z^1} H^2 \xrightarrow{L_z^2} H^3 \xrightarrow{L_z^3} \dots$$

Then the representations T'_z and T_z are equivalent.

Proof. Note that for every $n = 1, 2, \dots$ the subspace $\text{Cyl}^n \subset \mathcal{H} = L^2(\mathfrak{S}, \mu_t)$ of cylinder functions of level n is invariant with respect to the operators $T_z(g)$ where $g \in G(n)$. This follows from the definition of the representation T_z , and the fact that for all $g \in G(n)$ the function $x \mapsto c(x, g)$ is a cylinder function of level n :

$$c(x, g) = [p_n(x) \cdot g] - [p_n(x)], \quad x \in \mathfrak{S}, \quad g \in G(n).$$

Since the image of the measure μ_t with respect to the canonical projection $p_n: \mathfrak{S} \rightarrow S(n)$ coincides with μ_t^n , we can identify the Hilbert spaces $\text{Cyl}^n \subset \mathcal{H}$ and $L^2(S(n), \mu_t^n)$. The operators $T_z(g) \upharpoonright_{\text{Cyl}^n}$, where $g \in G(n)$, take the form

$$(T_z(g)f)(x) = f(x \cdot g) z^{[x \cdot g] - [x]} \quad (3-3-2)$$

(here $x \in S(n)$, $f \in L^2(S(n), \mu_t^n)$.)

Define a function F_z^n on the group $S(n)$ by the formula

$$F_z^n(x) = \left(\frac{n!}{t(t+1) \dots (t+n-1)} \right)^{1/2} z^{[x]}, \quad x \in S(n). \quad (3-3-3)$$

Then (3-3-2) can be written in the form

$$(T_z(g)f)(x) = f(x \cdot g) \frac{F_z^n(x \cdot g)}{F_z^n(x)}.$$

Note that the function $|F_z^n(x)|^2$ coincides with the density of the measure μ_t^n with respect to the Haar measure μ_1^n . It follows that the operator of multiplication by the function F_z^n defines an isometry

$$\text{Cyl}^n = L^2(S(n), \mu_t^n) \longrightarrow L^2(S(n), \mu_1^n) = H^n$$

intertwining the representations $T_z \upharpoonright_{\text{Cyl}^n}$ and Reg^n of the group $G(n)$.

Consider now the commutative diagram

$$\begin{array}{ccc} L^2(S(n), \mu_t^n) = \mathbb{C}^n & \longrightarrow & \mathbb{C}^{n+1} = L^2(S(n+1), \mu_t^{n+1}) \\ \downarrow & & \downarrow \\ H^n & \xrightarrow{\tilde{L}_z^n} & H^{n+1} \end{array}$$

where the top arrow denotes the natural embedding (lifting of functions via the projection $p_{n,n+1}$), the vertical arrows correspond to multiplication by F_z^n and

F_z^{n+1} , respectively, and the bottom arrow \tilde{L}_z^n is defined by the commutativity requirement. It follows that if $f \in H^n$ and $x \in S(n+1)$, then

$$\begin{aligned} (\tilde{L}_z^n f)(x) &= F_z^{n+1}(x) (F_z^n(p_n(x)))^{-1} f(p_n(x)) = \\ &= \sqrt{\frac{n+1}{t+n}} z^{[x]-[p_n(x)]} f(p_n(x)). \end{aligned}$$

Since

$$[x] = \begin{cases} [p_n(x)] + 1 & \text{if } x \in S(n) \subset S(n+1); \\ [p_n(x)] & \text{if } x \in S(n+1) \setminus S(n), \end{cases}$$

we conclude that $\tilde{L}_z^n = L_z^n$.

By the very definition, \tilde{L}_z^n is an isometric embedding which commutes with the action of $G(n)$. Hence, the inductive limit representation T'_z is well defined. From the commutative diagram above we conclude that T_z and T'_z are equivalent. \square

3.4. Representations T_0 and T_∞ .

Proposition 3.4.1.

(i) For every $n = 1, 2, \dots$, the isometry $L_z^n: H^n \rightarrow H^{n+1}$ admits a continuous continuation, with respect to the parameter $z \in \mathbb{C}^*$, to the points $z = 0$ and $z = \infty$ of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Therefore, the definition of the inductive limit representation T'_z also makes sense for the values $z = 0$ and $z = \infty$.

(ii) The representation T'_∞ is equivalent to the natural two-sided regular representation of the group $G = S(\infty) \times S(\infty)$ on the Hilbert space $l^2(S(\infty))$.

Proof. (i) Since $t = z\bar{z}$, (3-3-1) implies that the limits

$$L_0^n = \lim_{z \rightarrow 0} L_z^n \quad \text{and} \quad L_\infty^n = \lim_{|z| \rightarrow \infty} L_z^n$$

do exist, and have the form

$$\begin{aligned} (L_0^n f)(x) &= \begin{cases} 0, & \text{if } x \in S(n) \subset S(n+1), \\ \sqrt{\frac{n+1}{n}} f(p_n(x)), & \text{if } x \in S(n+1) \setminus S(n), \end{cases} \\ (L_\infty^n f)(x) &= \begin{cases} \sqrt{n+1} f(p_n(x)), & \text{if } x \in S(n) \subset S(n+1), \\ 0, & \text{if } x \in S(n+1) \setminus S(n). \end{cases} \end{aligned} \quad (3-4-1)$$

By continuity, L_0^n and L_∞^n determine isometric embeddings $H^n \rightarrow H^{n+1}$ commuting with the action of the group $G(n)$. Thus, we can use them to construct inductive limits T'_0, T'_∞ of the two-sided regular representations of the groups $S(n) \times S(n)$.

(ii) For every $n = 1, 2, \dots$ consider the map $l^2(S(n)) \rightarrow H^n$ defined as multiplication by the scalar $\sqrt{n!}$. This is an isometry, since the counting measure on $S(n)$ equals $n! \mu_1^n$. Under identification of both spaces by this map, the embedding L_∞^n turns into the natural embedding $l^2(S(n)) \rightarrow l^2(S(n+1))$. This completes the proof. \square

Using the identification $T_z = T'_z$ we now may extend the definition of representations T_z to the values $z = 0$ and $z = \infty$ of the parameter z . In this way we obtain a family of representations T_z parametrized by the points of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. We have shown that our family forms a continuous deformation of the standard two-sided regular representation of G in $l^2(S(\infty))$. This is a justification of the term “generalized regular representation”.

3.5. A construction of T_z via representations T_0 and T_∞ . Let us discuss now the formula for the isometric embeddings $L_z^n : H^n \rightarrow H^{n+1}$. We derived this formula from the initial definition of the representations T_z with $z \in \mathbb{C}^*$. Then, taking a limit transition in the formula, we completed the construction of the representations at the points $z = 0$ and $z = \infty$.

Here we aim to show that these two steps can be realized in opposite order. We start with the definition (3-4-1) of the embeddings L_0^n and L_∞^n , and then pass to the general operators L_z^n . First, we have to check that (3-4-1) indeed defines isometric embeddings, equivariant with respect to $G(n)$. For L_∞^n , this immediately follows from the basic property of the canonical projection. As for L_0^n , we observe that up to a scalar multiple, L_0^n can be defined as lifting along the fibers of the canonical projection $p_{n,n+1} : S(n+1) \rightarrow S(n)$, omitting the natural section $S(n) \rightarrow S(n+1)$. This implies equivariance. To prove the isometry property, we use the fact that the fiber over any point of $S(n)$ consists just of n points, except a single point belonging to the section.

Next, we note that the spaces $L_0^n(H^n)$ and $L_\infty^n(H^n)$ are mutually orthogonal subspaces of H^{n+1} : the functions in the second space are supported by the subgroup $S(n) \subset S(n+1)$, and those in the first space are supported by $S(n+1) \setminus S(n)$.

Comparing (3-3-1) and (3-4-1) we see that for $z \in \mathbb{C}^*$, L_z^n is a linear combination of L_0^n and L_∞^n :

$$L_z^n = \frac{\sqrt{n}}{\sqrt{t+n}} L_0^n + \frac{z}{\sqrt{t+n}} L_\infty^n. \quad (3-5-1)$$

Moreover, the coefficients of L_0^n and L_∞^n satisfy the relation

$$\left| \frac{\sqrt{n}}{\sqrt{t+n}} \right|^2 + \left| \frac{z}{\sqrt{t+n}} \right|^2 = \frac{n + z\bar{z}}{t+n} = 1.$$

It follows at once that the operator L_z^n defined by the formula (3-5-1) is a $G(n)$ -equivariant isometry.

Formula (3-5-1) looks very simple and natural. This is an argument in favor of “naturalness” of the representations T_z .

3.6. Connection between T_z and T_{-z} . For any $s \in S(\infty)$, the number $\text{inv}(s)$ of inversions in s is finite. Let

$$\text{sgn}(s) = (-1)^{\text{inv}(s)} = \pm 1.$$

Then $\text{sgn} : S(\infty) \rightarrow \{\pm 1\}$ is a (unique) nontrivial one-dimensional representation of the group $S(\infty)$.

Proposition 3.6.1. *For every $z \in \mathbb{C} \cup \{\infty\}$, T_{-z} is equivalent to $T_z \times (\text{sgn} \times \text{sgn})$.*

Proof. Given $x \in \mathfrak{S}$ and $g = (g_1, g_2) \in G$, let n be so large that $g \in G(n)$. Then

$$c(x, g) = [g_2^{-1} p_n(x) g_1] - [p_n(x)].$$

It follows that $c(x, g) \in 2\mathbb{Z}$ if the permutation $g_1 g_2^{-1}$ is even, and $c(x, g) \in 2\mathbb{Z} + 1$ if $g_1 g_2^{-1}$ is odd. Using the initial definition of T_z (for $z \neq 0, \infty$) we derive that

$$T_{-z}(g) = \text{sgn}(g_1 g_2^{-1}) T_z(g).$$

When $z = 0, \infty$, Proposition 3.6.1 claims the equivalence

$$T_0 \cong T_0 \otimes (\text{sgn} \times \text{sgn}), \quad T_\infty \cong T_\infty \otimes (\text{sgn} \times \text{sgn}).$$

Such an equivalence is indeed provided by the operator of multiplication by the function $\text{sgn}(\cdot)$ (we use the realization of the representations as inductive limits, see subsection 3.4). \square

4. THE DISTINGUISHED SPHERICAL FUNCTION

4.1. The distinguished vector and the coherent system M_z (case $z \neq 0, \infty$). Assume $z \in \mathbb{C} \setminus \{0\}$ and consider the generalized regular representation T_z . According to the initial construction of T_z in Hilbert space $\mathcal{H} = L^2(\mathfrak{S}, \mu_t)$ (see §3.1), T_z comes with a *distinguished vector* ξ_0 : this vector is simply the function $f_0 \equiv 1$ on the space \mathfrak{S} . Clearly, ξ_0 is K -invariant and has norm 1.

In the inductive limit realization of T_z as described in §3.3, the same vector ξ_0 is represented by the functions $F_z^n \in L^2(S(n), \mu_1^n)$ defined in (3-3-3).

Note that the whole space of K -invariant vectors in \mathcal{H} is infinite-dimensional. However, explicitly constructing invariant vectors other than the distinguished one is a nontrivial task.

Let φ_z denote the spherical function on the group G corresponding to the distinguished vector ξ_0 , let χ_z be the related character of the group $S(\infty)$, and let M_z be the corresponding coherent system. We will derive a nice expression for M_z . As for χ_z , it seems that it does not admit a simple explicit expression as a function on the symmetric groups $S(n)$. In other words, the Fourier coefficients of the functions $\chi_z|_{S(n)}$ are simple whereas the functions themselves are complex.

Recall a standard notation related to Young diagrams. For a particular box $b \in \lambda$ with coordinates (i, j) , the number

$$c(b) = j - i \quad \text{and}$$

is called the *content* of b .

Theorem 4.1.1. *Let $z \in \mathbb{C}^*$ and $t = z\bar{z}$. Consider the coherent system $M_z = \{M_z^{(n)}\}$ as defined above. For any Young diagram $\lambda \vdash n$,*

$$M_z(\lambda) = \frac{\prod_{b \in \lambda} |z + c(b)|^2}{t(t+1) \dots (t+n-1)} \frac{\dim^2 \lambda}{n!} \quad (4-1-1)$$

where we abbreviate

$$M_z(\lambda) = M_z^{(n)}(\lambda), \quad \lambda \in \mathbb{Y}_n.$$

Proof. It will be convenient to identify T_z with the inductive limit T'_z of regular representations Reg^n . Recall that in this realization the representation space is defined as the Hilbert completion H of the inductive limit of finite dimensional Hilbert spaces $H^n = L^2(S(n), \mu_1^n)$. The distinguished vector f_0 belongs to H^1 , hence to all of H^n . As an element of H^n it coincides with the function F_z^n introduced in §3. Therefore, for $s \in S(n)$

$$\chi_z|_{S(n)}(s) = (\text{Reg}^n(s, e) F_z^n, F_z^n) = \frac{1}{n!} \sum_{s_1 \in S(n)} F_z^n(s_1 s) \overline{F_z^n(s_1)}.$$

This can be rewritten as

$$\chi_z|_{S(n)} = (F_z^n)^* * F_z^n,$$

where $f^*(s) = \overline{f(s^{-1})}$ denotes the standard involution on the group algebra $\mathbb{C}[S(n)]$, and “ $*$ ” is the convolution product taken with respect to normalized Haar measure μ_1^n .

Note that F_z^n is a central function on $S(n)$, hence it can be decomposed as a sum of characters χ^λ ,

$$F_z^n = \sum_{\lambda \vdash n} a(\lambda) \chi^\lambda, \quad (4-1-2)$$

where $a(\lambda)$ are appropriate complex coefficients. By virtue of the orthogonality relations,

$$(\chi^\lambda)^* * \chi^\mu = \delta_{\lambda\mu} \frac{\chi^\lambda}{\dim \lambda}, \quad \lambda, \mu \vdash n,$$

hence

$$M_z(\lambda) = |a(\lambda)|^2. \quad (4-1-3)$$

Recall that

$$F_z^n(x) = \left(\frac{n!}{t(t+1) \dots (t+n-1)} \right)^{1/2} z^{[x]}, \quad (4-1-4)$$

where $[x]$ denotes the number of cycles of a permutation $x \in S(n)$, see §3.3. We are interested in the decomposition of the central function $z^{[x]}$.

Lemma 4.1.2. *Given $n = 1, 2, \dots$ and $z \in \mathbb{C}^*$, the decomposition of the central function $x \mapsto z^{[x]}$ on the group $S(n)$ along the characters χ^λ , $\lambda \vdash n$, can be written in the form*

$$\begin{aligned} z^{[x]} &= \sum_{\lambda \vdash n} \prod_{b \in \lambda} (z + c(b)) \cdot \frac{\dim \lambda}{n!} \chi^\lambda(x) = \\ &= \sum_{\lambda \vdash n} \prod_{b \in \lambda} \frac{z + c(b)}{h(b)} \chi^\lambda(x). \end{aligned} \quad (4-1-5)$$

Keeping together (4-1-2), (4-1-3), (4-1-4), (4-1-5) we get the desired formula (4-1-1). Thus, it remains to prove the lemma.

Proof of the lemma. We switch from central functions on the group $S(n)$ to symmetric functions. This is done using the classical *characteristic map* “ch” establishing a bijection between central functions on $S(n)$ and homogeneous symmetric functions of degree n , see [Mac, 1.7]. It is well known that $\text{ch}(\chi^\lambda) = s_\lambda$, hence we have to prove the formula

$$\text{ch}(z^{[\cdot]}) = \sum_{\lambda \vdash n} \prod_{b \in \lambda} \frac{z + c(b)}{h(b)} \cdot s_\lambda, \quad (4-1-6)$$

where s_λ are the Schur functions. Let us recall the definition of ch. If F is a central function on $S(n)$, $\rho = (1^{k_1} 2^{k_2} \dots)$ is a partition of n , and $x_\rho \in S(n)$ is a permutation of cycle type ρ , then

$$\text{ch } F = \sum_{\rho \vdash n} z_\rho^{-1} F(x_\rho) p_\rho.$$

Here z_ρ is the order of the centralizer of x_ρ ,

$$z_\rho = \frac{n!}{1^{k_1} 2^{k_2} \dots k_1! k_2! \dots},$$

and $p_\rho = p_1^{k_1} p_2^{k_2} \dots$ are the monomials in the power sums p_1, p_2, \dots . Note that $[x_\rho] = k_1 + k_2 + \dots$ and $k_1 + 2k_2 + 3k_3 + \dots = n$.

Denote by y_1, y_2, \dots a sequence of formal variables of symmetric functions, and let u be still another formal variable. One can write

$$\begin{aligned} 1 + \sum_{n \geq 1} \text{ch}(z^{[\cdot]}) u^n &= \sum_{(1^{k_1} 2^{k_2} \dots)} \frac{z^{k_1+k_2+\dots} u^{1k_1+2k_2+\dots}}{1^{k_1} 2^{k_2} \dots k_1! k_2! \dots} p_1^{k_1} p_2^{k_2} \dots = \\ &= \prod_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{z^k p^k u^{nk}}{n^k k!} = \exp z \left(\frac{up_1}{1} + \frac{u^2 p_2}{2} + \dots \right) = \\ &= \exp z \sum_{i=1}^{\infty} \left(\frac{uy_i}{1} + \frac{(uy_i)^2}{2} + \dots \right) = \\ &= \exp \left(-z \sum_{i=1}^{\infty} \ln(1 - uy_i) \right) = \prod_{i=1}^{\infty} (1 - uy_i)^{-z}. \end{aligned}$$

The formula (4-1-6) takes the form

$$\prod_{i=1}^{\infty} (1 - uy_i)^{-z} = \sum_{\lambda} \prod_{b \in \lambda} \frac{z + c(b)}{h(b)} \cdot s_{\lambda}(uy_1, uy_2, \dots),$$

where λ in the right hand side runs over all Young diagrams. Replacing uy_i with y_i , we arrive at the identity

$$\prod_{i=1}^{\infty} (1 - y_i)^{-z} = \sum_{\lambda} \prod_{b \in \lambda} \frac{z + c(b)}{h(b)} \cdot s_{\lambda}(y_1, y_2, \dots). \quad (4-1-7)$$

Recall that the coefficients of Schur functions in the right hand side are the polynomials in z , hence it suffices to prove (4-1-7) for $z = N = 1, 2, \dots$.

It is well known ([Mac, I.3, Example 4]) that

$$\prod_{b \in \lambda} \frac{N + c(b)}{h(b)} = s_{\lambda}(\underbrace{1, \dots, 1}_N)$$

(this is the dimension of the irreducible representation of the group $GL(\mathbb{C})$ with the highest weight $(\lambda_1, \dots, \lambda_N)$ if $\lambda_{N+1} = \lambda_{N+2} = \dots = 0$, and 0 otherwise). The formula (4-1-3) takes the form

$$\prod_{i=1}^{\infty} (1 - y_i)^{-N} = \sum_{\lambda} s_{\lambda}(\underbrace{1, \dots, 1}_N) s_{\lambda}(y_1, y_2, \dots).$$

This is a specialization of a more general identity ([Mac, Ch. I, (4.3)])

$$\prod_{j=1}^{\infty} \prod_{i=1}^{\infty} (1 - u_j y_i)^{-1} = \sum_{\lambda} s_{\lambda}(u_1, u_2, \dots) s_{\lambda}(y_1, y_2, \dots)$$

where we put $u_1 = \dots = u_N = 1$ and $u_{N+1} = u_{N+2} = \dots = 0$. This completes the proof of Lemma 4.1.2 and Theorem 4.1.1. \square

4.2. The limit coherent systems M_0 and M_∞ . In Theorem 4.1.1 we did not consider the parameter values $z = 0$ and $z = \infty$. However, one can see from (4-1-1) that there exist the limits

$$M_0 = \lim_{z \rightarrow 0} M_z, \quad M_\infty = \lim_{z \rightarrow \infty} M_z,$$

which are also coherent systems on the Young lattice. The coherent system M_0 is supported by hook diagrams only (see Proposition 4.3.1 (iii) below), and

$$M_\infty^{(n)}(\lambda) = \frac{\dim^2 \lambda}{n!}, \quad \lambda \in \mathbb{Y}_n, \quad (4-2-1)$$

is the so-called *Plancherel measure*.

According to Proposition 9.5.1 these limiting coherent systems give rise to certain characters χ_0 and χ_∞ of the group $S(\infty)$, and to certain spherical functions φ_0 and φ_∞ .

Proposition 4.2.1. *The functions φ_0 , φ_∞ are spherical functions of the representations $T_0 = T'_0$ and $T_\infty = T'_\infty$ respectively. That is, they coincide with matrix coefficients of certain K -invariant vectors of the representations in question.*

Proof. In order to see this, examine the behavior of the distinguished vector ξ_0 (recall that as an element of the space H^n it coincides with the function F_z^n) as long as $z \rightarrow 0$ or $z \rightarrow \infty$.

Set $z = r\zeta$ where $r > 0$ and ζ is a point of the unit circle $S^1 \subset \mathbb{C}^*$. If ζ is fixed, the limits

$$F_0^n = \lim_{r \rightarrow 0} F_{r\zeta}^n, \quad F_\infty^n = \lim_{r \rightarrow \infty} F_{r\zeta}^n, \quad n = 1, 2, \dots$$

exist and have the form

$$F_0^n(x) = \begin{cases} \sqrt{n} \zeta & \text{if } [x] = 1, \text{ i.e., if } x \text{ is a cycle in } S(n) \text{ of maximal length } n, \\ 0 & \text{otherwise;} \end{cases}$$

$$F_\infty^n(x) = \begin{cases} \sqrt{n!} \zeta^n & \text{if } x = e, \\ 0 & \text{if } x \neq e. \end{cases}$$

In these expressions, ζ enters as a scalar factor only. Hence the corresponding spherical functions do not depend on the choice of ζ . Clearly, they coincide with φ_0 and φ_∞ , respectively. \square

Thus, our definition of the distinguished vector ξ_0 (§4.1) can be extended to the limit cases $z = 0$ and $z = \infty$ — at least, up to an unessential scalar factor. Note that the representation T_∞ can be realized in the Hilbert space $\ell^2(S(\infty))$, and then ξ_0 can be identified with the delta function at $e \in S(\infty)$.

Note also that φ_∞ is simply the characteristic function of the subgroup $K \subset G$, and χ_∞ is the delta function at the identity element of $S(\infty)$.

We proceed to analysis of the formula (4-1-1).

4.3. Support of M_z . Here we consider the coherent system M_z with arbitrary $z \in \mathbb{C} \cup \{\infty\}$. The definition of the support of a coherent system is given in §9.4.

Proposition 4.3.1. (i) If $z \notin \mathbb{Z}$, then $\text{supp}(M_z)$ is the whole set \mathbb{Y} .

(ii) If λ is a nonzero integer, $\lambda = k$ or $\lambda = -k$, where $k = 1, 2, \dots$, then $\text{supp}(M_z)$ consists of those Young diagrams λ that have no more than k rows, or, respectively, no more than k columns.

(iii) If $z = 0$, then $\text{supp}(M_z)$ is the set of hook diagrams, i.e., Young diagrams contained inside the union of the first row and the first column.

Proof. (i) If $z \notin \mathbb{Z} \cup \{\infty\}$, then the numerator in (4-1-1) does not vanish for all λ , hence $M_z(\lambda) \neq 0$. If $z = \infty$, then it follows from (4-2-1) that $M_\infty(\lambda) \neq 0$.

(ii) Assume that $z = k$. Then the zeros in the numerator of (4-1-1) correspond to the boxes $b = (i, j) \in \lambda$ such that $c(b) = j - i = -k$. These boxes lie on a diagonal of λ passing through the box $(k+1, 1)$ in the first column. The lack of such boxes in λ is clearly equivalent to the fact that λ contains k or less rows. In a similar fashion, if $z = -k$ then $-k + c(b) \neq 0$ for all $b \in \lambda$ if and only if λ contains no more than k columns.

(iii) Assume that $z = 0$ and consider the limit $M_0 = \lim_{z \rightarrow 0} M_z$. When $z \rightarrow 0$, the zero factor $t = z\bar{z}$ in the denominator of (4-3-1) cancels with the factor in the numerator corresponding to the box $b = (1, 1)$ (every nonempty diagram $\lambda \neq \emptyset$ contains this box). Other zero factors in the numerator correspond to the boxes $(2, 2), (3, 3), \dots$ on the main diagonal. The absence of such boxes just means that λ is a hook. \square

If a hook diagram λ has *arm length* $a = \lambda_1 - 1$ and *leg length* $l = \lambda'_1 - 1$, then

$$M_0(\lambda) = \frac{|z - l|^2 \dots |z - 1|^2 |z + 1|^2 \dots |z + a|^2}{(t+1)(t+2)(t+3) \dots (t+n-1)} \frac{\dim^2 \lambda}{n!}.$$

4.4. When the distinguished vector ξ_0 is cyclic.

Proposition 4.4.1. Assume $z \notin \mathbb{Z}$, then the distinguished vector ξ_0 is a cyclic vector of the representation T_z .

Proof. When $z = \infty$, this is evident from the realization in the space $\ell^2(S(\infty))$. Assume now $z \in \mathbb{C} \setminus \mathbb{Z}$.

Since T_z is the inductive limit of regular representations Reg^n and ξ_0 belongs to the spaces H^n of all those representations, it suffices to check that ξ_0 is cyclic in Reg^n for all n . Recall that

$$\text{Reg}^n \cong \bigoplus_{\lambda \vdash n} (\pi^\lambda \times \pi^\lambda).$$

Each of the irreducible representations $\pi^\lambda \times \pi^\lambda = \pi^\lambda \times (\pi^\lambda)^*$ of the group $G(n) = S(n) \times S(n)$ is spherical with respect to diagonal subgroup $K(n)$, and the corresponding spherical vector is the function χ^λ . Since $\xi_0 = F_z^n \in H^n$ is a $K(n)$ -invariant vector, too, it is cyclic if and only if all of its coefficients in the decomposition in functions χ^λ are nonzero. But $M_z(\lambda)$ is the square modulus of the coefficient of χ^λ , hence the claim follows from Proposition 4.3.1(i). \square

We shall see below that ξ_0 is *not* cyclic in T_z if $z \in \mathbb{Z}$.

4.5. The equivalence of representations T_z and $T_{\bar{z}}$.

Proposition 4.5.1. *The representations T_z and $T_{\bar{z}}$ are unitarily equivalent for all $z \in \mathbb{C} \cup \{\infty\}$.*

Proof. If $z \in \mathbb{R}$ or $z = \infty$, there is nothing to prove. Hence, we may assume that $z \notin \mathbb{R} \cup \{\infty\}$, in particular, $z \notin \mathbb{Z}$. By virtue of Proposition 4.6.1, it suffices to check that $\varphi_z = \varphi_{\bar{z}}$ which is equivalent to $\chi_z = \chi_{\bar{z}}$ and to $M_z = M_{\bar{z}}$. But this last formula follows directly from (4-1-1). \square

Note that it is not immediate from the definition that T_z and $T_{\bar{z}}$ are equivalent.

Proposition 4.5.2. *Assume that $z \in \mathbb{C} \setminus \mathbb{Z}$.*

(i) *There exists a unique operator A_z intertwining representation T_z and $T_{\bar{z}}$, and identifying their distinguished spherical vectors.*

(ii) *Realize the representations $T_z, T_{\bar{z}}$ as inductive limits of representations Reg^n . Then the operator A_z preserves the subspaces H^n , hence determines an operator $A_{n,z}: H^n \rightarrow H^n$ commuting with the representation Reg^n , for all $n = 1, 2, \dots$.*

(iii) *The operator $A_{n,z}$ on the space $H^n = L^2(S(n), \mu_1^n)$ is the convolution operator with the central function*

$$\Theta_{n,z} = \sum_{\lambda \vdash n} \left(\prod_{b \in \lambda} \frac{\bar{z} + c(b)}{z + c(b)} \right) \dim \lambda \cdot \chi^\lambda.$$

Proof. (i) Follows from the fact that the distinguished vectors are cyclic (for $z \notin \mathbb{Z}$), and from the coincidence of the corresponding spherical functions $\varphi_z, \varphi_{\bar{z}}$.

(ii) Follows from (i) and the fact that the distinguished vector (for $z \notin \mathbb{Z}$) is a $G(n)$ -cyclic vector in the representation Reg^n , for all $n = 1, 2, \dots$.

(iii) We have to find the operator in $L^2(S(n), \mu_1^n)$ that commutes with the regular representation Reg^n , and transforms the function

$$F_z^n = C \sum_{\lambda \vdash n} \left(\prod_{b \in \lambda} \frac{z + c(b)}{h(b)} \right) \chi^\lambda$$

into

$$F_{\bar{z}}^n = C \sum_{\lambda \vdash n} \left(\prod_{b \in \lambda} \frac{\bar{z} + c(b)}{h(b)} \right) \chi^\lambda,$$

with the same factor $C \neq 0$ (we do not need its precise form at the moment). Every operator commuting with Reg^n is a convolution operator with some central function

$$\Theta = \sum_{\lambda \vdash n} \theta(\lambda) \chi^\lambda.$$

Note that the convolution operator with the function $\dim \lambda \cdot \chi^\lambda$ is the projection onto the irreducible component $\pi^\lambda \times \pi^\lambda$ of the representation Reg^n . It follows that $\Theta = \Theta_{n,z}$. \square

4.6. Reducibility of representations T_z . For $z \in \mathbb{C} \cup \{\infty\}$, let \tilde{T}_z denote the subrepresentation in T_z realized in the cyclic span of the distinguished vector. In other words, \tilde{T}_z is the cyclic unitary representation of the group G generated by the positive definite function φ_z . If $z \notin \mathbb{Z}$, then \tilde{T}_z coincides with T_z by Proposition

4.4.1; we shall see below that for $z \in \mathbb{Z}$ it is a proper subrepresentation of T_z . Note that

$$\varphi_1(g) \equiv 1, \quad \varphi_{-1}(g) \equiv \text{sgn}(g_1 g_2^{-1}), \quad g = (g_1, g_2) \in G,$$

so that for $z = \pm 1$ our representation \tilde{T}_z is one-dimensional (more precisely, trivial for $z = 1$ and equivalent to $\text{sgn} \times \text{sgn}$ for $z = -1$). Moreover, for $z = \infty$ the representation \tilde{T}_∞ is irreducible, since the two-sided regular representation T_∞ of the group $G = S \times S$ in $l^2(S)$ is irreducible. We shall show now that in all other cases our cyclic representation \tilde{T}_z (hence the entire representation T_z) is reducible.

Proposition 4.6.1. *For every $z \in \mathbb{C} \setminus \{\pm 1\}$ the cyclic representation $\tilde{T}_z \subseteq T_z$ generated by the distinguished vector is reducible.*

Proof. Let χ_z denote the character corresponding to the spherical function φ_z . Irreducibility of the representation \tilde{T}_z would imply that χ_z is an extreme character and hence coincides with a certain character $\chi^{\alpha\beta}$ from the Thoma list (see the Appendix). In this case we would also have the equality $M_z = M^{(\alpha,\beta)}$ of the corresponding coherent systems on the Young lattice. We shall compare the values of M_z and $M^{(\alpha,\beta)}$ on one-row diagrams $\lambda = (n)$, $n = 1, 2, \dots$ and derive that the equality only holds for $z = \pm 1, \infty$.

Indeed, it follows from (4-3-1) that the generating function for $M_z((n))$, $z \neq 0, \infty$, is

$$1 + \sum_{n \geq 1} M_z((n)) w^n = {}_2F_1(z, \bar{z}; z\bar{z}; w),$$

where w is a parameter. On the other hand, the Thoma formula implies that

$$1 + \sum_{n \geq 1} M^{\alpha\beta}((n)) w^n = e^{\gamma w} \prod_i \frac{1 + \beta_i w}{1 - \alpha_i w}, \quad \gamma = 1 - \sum_{k=1}^{\infty} (\alpha_k + \beta_k).$$

Hence, we are led to study the possibility of the equality

$${}_2F_1(z, \bar{z}; z\bar{z}; w) = e^{\gamma w} \prod_i \frac{1 + \beta_i w}{1 - \alpha_i w}. \quad (4-6-1)$$

This formula would also imply that

$${}_2F_1(-z, -\bar{z}; z\bar{z}; w) = e^{\gamma w} \prod_i \frac{1 + \alpha_i w}{1 - \beta_i w}, \quad (4-6-2)$$

since the tensor multiplication by the nontrivial one-dimensional representation $\text{sgn} \times \text{sgn}$ switches z to $-z$ (by Proposition 3.6.1) and replaces $\chi^{\alpha\beta}$ with $\chi^{\beta\alpha}$.

It is now easy to see that the equalities (4-8-1), (4-6-2) are only possible if $z = \pm 1$. In fact, the hypergeometric series converges absolutely in the open disk $|w| < 1$, hence the right hand side of (4-6-1), (4-6-2) cannot have poles in this disk. Recall that all the Thoma parameters are positive, so that there are no cancellations between numerators and denominators. Therefore, $\alpha_1 = 1$ or $\beta_1 = 1$ or $\gamma = 1$, and all other parameters vanish. The first case corresponds to $z = 1$, the second one to $z = -1$, and the last one corresponds to $z = \infty$ and cannot occur for a finite z .

It remains to consider the case $z = 0$. In the limit $z \rightarrow 0$ the generating function takes the form $1 + \sum_{n \geq 1} \frac{1}{n} w^n$ and the equalities (4-6-1), (4-6-2) cannot hold in this case, too. \square

4.7. Transition probabilities. Recall from Proposition 4.5.1 that in case $z \notin \mathbb{Z}$ the support of M_z is the entire Young graph. If $z = \pm k$ where $k = 1, 2, \dots$, the support is made of the Young diagrams with k or less rows (columns). If $z = 0$, then M_z is supported by the hook diagrams.

Let $p_z(\lambda, \nu)$ denote the transition probabilities of the coherent system M_z , see §9... These quantities are defined for any $\lambda \in \text{supp}(M_z)$.

Proposition 4.7.1. *Let $\lambda \in \mathbb{Y}_n$, $\nu \in \mathbb{Y}_{n+1}$, $\lambda \nearrow \nu$. We have*

$$p_\infty(\lambda, \nu) = \frac{\dim \nu}{(n+1) \dim \lambda},$$

$$p_z(\lambda, \nu) = \frac{|z + c_{\lambda\nu}|^2}{z\bar{z} + n} p_\infty(\lambda, \nu), \quad \lambda \in \mathbb{Y}_n \cap \text{supp}(M_z), \quad z \neq \infty$$

where $c_{\lambda\nu} = c(\nu \setminus \lambda)$ is the content of the box $\nu \setminus \lambda$.

Proof. Follows immediately from (4-1-1) and the definition of transition probabilities. \square

5. THE COMMUTANT AND BLOCK DECOMPOSITION

5.1. Simplicity of spectrum. Let T_z , $z \in \mathbb{C} \cup \{\infty\}$ be a generalized regular representation. We shall work with the realization of T_z as inductive limit of the two-sided regular representation Reg^n of the group $G(n) = S(n) \times S(n)$. As before, we denote the space of the representation Reg^n by H^n . The representation T_z acts in the Hilbert completion H of the space $\bigcup_{n \geq 1} H^n$, where the maps $L_z^n: H^n \rightarrow H^{n+1}$ (depending on z) were introduced in §3. It will be important that L_z^n depends continuously on the parameter z ranging over the Riemann sphere. Recall that Reg^n is the direct sum of irreducible representations $\pi^\lambda \times \pi^\lambda$, $\lambda \in \mathbb{Y}_n$ of the group $G(n)$. There are no multiple components in this decomposition.

Denote by $P_n: H \rightarrow H^n$ the orthogonal projection onto H^n , by $H(\lambda) \subset H^n$ the space of the representation $\pi^\lambda \times \pi^\lambda$, and by $P(\lambda): H \rightarrow H(\lambda)$ the orthogonal projection onto $H(\lambda)$. Note that the projectors $P(\lambda)$, $\lambda \in \mathbb{Y}_n$, are pairwise orthogonal, and their sum equals P_n .

Let \mathcal{A} be the commutant of T_z , i.e., the algebra of all bounded operators in H commuting with the representation T_z . We know that for $z \notin \mathbb{Z}$ the representation T_z admits a cyclic K -invariant vector (the distinguished vector). On the other hand, (G, K) is a Gelfand pair [Ol3, §1]. It follows that for $z \notin \mathbb{Z}$, the algebra \mathcal{A} is isomorphic to the commutant of a commutative operator $*$ -algebra admitting a cyclic vector, whence \mathcal{A} is commutative. We shall presently give another proof of this fact, applicable for all z .

Proposition 5.1.1. *For every $z \in \mathbb{C} \cup \{\infty\}$ the commutant \mathcal{A} of the representation T_z is a commutative algebra.*

Proof. We have to prove that $AB = BA$ for arbitrary $A, B \in \mathcal{A}$. For every n , the operators $P_n A P_n$ and $P_n B P_n$ viewed as operators in the space H^n commute with the representation Reg^n . Since Reg^n multiplicity free, its commutant is commutative. Therefore,

$$P_n A P_n B P_n = P_n B P_n A P_n \quad \text{for all } n = 1, 2, \dots \quad (5-1-1)$$

Since $H^n \subset H$ form an increasing chain of subspaces and their union is dense in H , the projectors P_n converge to 1 strongly as $n \rightarrow \infty$. Moreover, the multiplication operation is continuous in the strong operator topology on every operator ball. Since the norms of all operators in (5-1-1) do not exceed the maximum of the numbers 1, $\|A\|$, $\|B\|$, we can pass to the limit in (5-1-1), which gives $AB = BA$. \square

Corollary 5.1.2. *For every $z \in \mathbb{C}$, the representation T_z is decomposable in a multiplicity free direct integral of admissible irreducible representations of the group G .*

Proof. The existence of a decomposition into a multiplicity free integral of irreducible representations follows from the fact that the group G is countable and the commutant is commutative. The admissibility is easily checked as in [Ol1, Theorem 3.6]. \square

5.2. The transition function of T_z . In order to simplify the notation, we identify the isometric embedding $L_z^n: H^n \rightarrow H^{n+1}$ with the partially isometric operator $L_z^n P_n$ in the space H . For every $\lambda \in \mathbb{Y}_n$, $\nu \in \mathbb{Y}_{n+1}$ (where $\lambda \nearrow \nu$) fix an isometric embedding $E(\lambda, \nu): H(\lambda) \rightarrow H(\nu)$ commuting with the action of the group $G(n)$. The choice of $E(\lambda, \nu)$ is unique, up to a complex factor of modulus 1. We identify $E(\lambda, \nu)$ with the partially isometric operator $E(\lambda, \nu)P(\lambda)$ acting in the whole space H .

For each pair of Young diagrams $\lambda \in \mathbb{Y}_n$, $\nu \in \mathbb{Y}_{n+1}$ consider the operator $P(\nu) L_z^n P(\lambda)$. This operator intertwines representations $\pi^\lambda \times \pi^\lambda$ and $(\pi_\nu \times \pi_\nu)|_{G(n)}$, hence is 0 unless $\lambda \nearrow \nu$. In the latter case it is proportional to $E(\lambda, \nu)$:

$$P(\nu) L_z^n P(\lambda) = \alpha_z(\lambda, \nu) E(\lambda, \nu), \quad \lambda \nearrow \nu.$$

Set

$$\tilde{p}_z(\lambda, \nu) = |\alpha_z(\lambda, \nu)|^2, \quad \lambda \nearrow \nu.$$

Clearly, this function does not depend on the choice of $E(\lambda, \nu)$. It is also clear that for any $\xi \in H(\lambda)$

$$\|P(\nu) \xi\|^2 = \begin{cases} \tilde{p}_z(\lambda, \nu) \|\xi\|^2, & \text{if } \lambda \nearrow \nu \\ 0, & \text{otherwise.} \end{cases}$$

Since the projections $P(\nu)$, $\nu \in \mathbb{Y}_{n+1}$ are pairwise orthogonal and sum up to P_{n+1} , it follows that

$$\sum_{\nu \searrow \lambda} \tilde{p}_z(\lambda, \nu) = 1 \quad \forall \lambda \in \mathbb{Y}_n, \quad n = 1, 2, \dots$$

We shall call $\tilde{p}_z(\lambda, \nu)$ the *transition function of the representation T_z* . In Theorem 5.5.1 below we show that it coincides with the transition probabilities of the coherent system M_z .

Let us emphasize that the transition function $\tilde{p}_z(\lambda, \nu)$ is defined on the edges of the graph $\mathbb{Y} \setminus \{\emptyset\}$, not the whole Young graph. That is, we *do not* attempt to define the value of this function when λ is the empty diagram \emptyset and ν is the one-box diagram.

5.3. The commutant in terms of the transition function. We shall presently show that the transition function of the representation T_z determines its commutant completely.

Let $\tilde{\mathcal{A}}$ denote the space of all bounded complex functions $A(\lambda)$ on the set of vertices of the graph $\mathbb{Y} \setminus \{\emptyset\}$ satisfying the condition

$$A(\lambda) = \sum_{\nu \searrow \lambda} \tilde{p}_z(\lambda, \nu) A(\nu), \quad \text{for all } \lambda \in \mathbb{Y}, \lambda \neq \emptyset, \quad (5-3-1)$$

where $\nu \searrow \lambda$ means $\lambda \nearrow \nu$. We consider $\tilde{\mathcal{A}}$ as a Banach space with the norm $\|A\| = \sup_{\lambda} |A(\lambda)|$.

Proposition 5.3.1. *The commutant \mathcal{A} of the generalized regular representation T_z considered as a Banach space with the ordinary operator norm is naturally isometric to the space $\tilde{\mathcal{A}}$.*

Proof. For every operator $A \in \mathcal{A}$ and every $n = 1, 2, \dots$ we have

$$P_n A P_n = \sum_{\lambda, \mu \in \mathbb{Y}_n} P(\lambda) A P(\mu).$$

But $P(\lambda) A P(\mu)$ intertwines the representations $\pi^\mu \times \pi^\mu$ and $\pi^\lambda \times \pi^\lambda$, hence can be nonzero only if $\lambda = \mu$. In this case the operator $P(\lambda) A P(\lambda)$ has to be proportional to $P(\lambda)$. Denoting the coefficient by $A(\lambda)$ we obtain

$$P_n A P_n = \sum_{\lambda \in \mathbb{Y}_n} A(\lambda) P(\lambda).$$

It is clear that

$$\|P_n A P_n\| = \sup_{\lambda \in \mathbb{Y}_n} |A(\lambda)|$$

which implies that

$$\|A\| = \sup_n \|P_n A P_n\| = \sup_{\lambda \in \mathbb{Y}} |A(\lambda)| = \|A(\cdot)\|,$$

where $\|A(\cdot)\|$ denotes the sup-norm of the function $A(\lambda)$.

Let us check now that for any $\lambda \in \mathbb{Y}_n$, $\nu \in \mathbb{Y}_{n+1}$,

$$P(\lambda) P(\nu) P(\lambda) = \begin{cases} \tilde{p}_z(\lambda, \nu) P(\lambda), & \text{if } \lambda \nearrow \nu, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, the operator $P(\nu)$ commutes with the action of $G(n) \subset G(n+1)$, hence the operator $P(\lambda) P(\nu) P(\lambda)$ which commutes with the irreducible representation $\pi^\lambda \times \pi^\lambda$ of the group $G(n)$ should be proportional to $P(\lambda)$. In order to find the coefficient we remark that for every $\xi \in H(\lambda)$ we have $P(\lambda) \xi = \xi$, so that

$$\begin{aligned} (P(\lambda) P(\nu) P(\lambda) \xi, \xi) &= (P(\nu) \xi, \xi) = (P(\nu) \xi, P(\nu) \xi) = \\ &= \begin{cases} \tilde{p}_z(\lambda, \nu) (\xi, \xi), & \text{if } \lambda \nearrow \nu, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

by the definition of the transition function. On the other hand, $(P(\lambda)\xi, \xi) = (\xi, \xi)$, so that the above coefficient equals $\tilde{p}_z(\lambda, \nu)$ if $\lambda \nearrow \nu$, and 0 otherwise.

Note now that $P(\lambda)P(\nu)P(\mu) = 0$ if $\lambda, \mu \in \mathbb{Y}_n$, $\mu \neq \lambda$. Therefore,

$$P_n P(\nu) P_n = \sum_{\lambda \nearrow \nu} \tilde{p}_z(\lambda, \nu) P(\lambda) \quad \forall \nu \in \mathbb{Y}_{n+1}.$$

Setting

$$A_n = P_n A P_n, \quad n = 1, 2, \dots$$

we see that the property (5-3-1) of the function $A(\cdot)$ simply means that

$$A_n = P_n A_{n+1} P_n, \quad n = 1, 2, \dots$$

Thus, we have constructed above an isometric embedding of \mathcal{A} into the space $\tilde{\mathcal{A}}$. In the opposite direction, we shall show that every function $A(\cdot) \in \tilde{\mathcal{A}}$ stems from some operator $A \in \mathcal{A}$. To this end we set

$$A_n = \sum_{\lambda \in \mathbb{Y}_n} A(\lambda) P(\lambda), \quad n = 1, 2, \dots$$

The condition (5-3-1) then implies that

$$A_n = P_n A_{n+1} P_n, \quad n = 1, 2, \dots,$$

and the condition $\|A(\cdot)\| < \infty$ implies

$$\sup_n \|A_n\| = \|A(\cdot)\| < \infty.$$

It follows that there exists a bounded operator

$$A = \text{w-lim}_{n \rightarrow \infty} A_n$$

where w-lim denotes the limit in the weak operator topology. Since A_n commutes with the action of the group $G(n)$, the operator A belongs to the commutant. It is clear that A_n coincides with $P_n A P_n$ for all n , hence our function $A(\cdot)$ corresponds to this very operator. \square

5.4. The multiplication in the space $\tilde{\mathcal{A}}$. Let us denote by $(A \circ B)(\cdot)$ the multiplication operation of functions $A(\cdot)$, $B(\cdot)$ in $\tilde{\mathcal{A}}$ corresponding to the operator multiplication in \mathcal{A} . Unfortunately there is no simple formula for this operation. Nevertheless, it can be described in terms of the transition function using an appropriate limit procedure.

It will be convenient to extend the definition of the transition function $\tilde{p}_z(\lambda, \nu)$ to all pairs of Young diagrams λ, ν . Given $\lambda \in \mathbb{Y}_n$, $\nu \in \mathbb{Y}_N$, we denote by $\mathcal{T}(\lambda, \nu)$ the set of paths

$$\tau = (\tau_n \nearrow \tau_{n+1} \nearrow \dots \nearrow \tau_N), \quad \tau_n = \lambda, \quad \tau_N = \nu,$$

from λ to ν in the Young graph (such paths are commonly called *skew Young tableaux* of shape ν/λ). Let

$$\tilde{p}_z(\tau) = \prod_{k=n+1}^N \tilde{p}_z(\tau_{k-1}, \tau_k)$$

denote the transition probability along the path τ , and let

$$\tilde{p}_z(\lambda, \nu) = \sum_{\tau \in \mathcal{T}(\lambda, \nu)} \tilde{p}_z(\tau)$$

be the total probability of the transition from λ to ν . If λ is not contained in ν , then the set $\mathcal{T}(\lambda, \nu)$ is empty and $\tilde{p}_z(\lambda, \nu) = 0$. If $\lambda \nearrow \nu$, then the definition of $\tilde{p}_z(\lambda, \nu)$ does not change.

Let $C(\mathbb{Y}_n)$ denote the algebra of functions on the finite set \mathbb{Y}_n , with pointwise multiplication. Given $n < N$, define a linear map

$$\alpha_{N,n} : C(\mathbb{Y}_N) \rightarrow C(\mathbb{Y}_n)$$

as follows: if $A(\cdot) \in C(\mathbb{Y}_N)$, then

$$(\alpha_{N,n}(A))(\lambda) = \sum_{\nu \in \mathbb{Y}_N} \tilde{p}_z(\lambda, \nu) A(\nu).$$

Proposition 5.4.1. *Assume that $A(\cdot), B(\cdot) \in \tilde{\mathcal{A}}$, and let*

$$A_N(\cdot) = A(\cdot) |_{\mathbb{Y}_N}, \quad B_N(\cdot) = B(\cdot) |_{\mathbb{Y}_N}$$

denote their restrictions to \mathbb{Y}_N . Then

$$(A \circ B)(\lambda) = \lim_{N \rightarrow \infty} \alpha_{N,n}(A_N(\cdot) B_N(\cdot))(\lambda), \quad \lambda \in \mathbb{Y}_n, \quad (5-4-1)$$

Proof. Denote by $A, B \in \mathcal{A}$ the operators corresponding to $A(\cdot), B(\cdot)$ and set $A_N = P_N A P_N, B_N = P_N B P_N$. Then

$$A_N B_N = \sum_{\nu \in \mathbb{Y}_N} A_N(\nu) B_N(\nu) P(\nu),$$

hence

$$P(\lambda) A_N B_N P(\lambda) = \alpha_{N,n}(A_N(\cdot) B_N(\cdot))(\lambda)$$

for all $\lambda \in \mathbb{Y}_n$. On the other hand, $A_N B_N$ converges strongly to AB as $N \rightarrow \infty$, so that the left hand side converges strongly to

$$P(\lambda) A B P(\lambda) = (A \circ B)(\lambda) P(\lambda),$$

which proves (5-4-1). \square

5.5. The identity of the functions p_z and \tilde{p}_z .

Theorem 5.5.1. *The transition function of the representation T_z , $z \in \mathbb{C} \cup \{\infty\}$, is given by the same expression as the transition probabilities of the distribution M_z , i.e.,*

$$\tilde{p}_z(\lambda, \nu) = \frac{|z + c_{\lambda\nu}|^2}{|z|^2 + n} \cdot \frac{\dim \nu}{(n+1) \dim \lambda}, \quad (5-5-1)$$

where $\lambda \in \mathbb{Y}_n$, $\nu \in \mathbb{Y}_{n+1}$, $\lambda \nearrow \nu$ and $c_{\lambda\nu} = c(\nu \setminus \lambda)$.

Remark 5.5.2. If $z \in \mathbb{Z}$, the function $p_z(\lambda, \nu)$ is formally defined on the edges of the proper subgraph $\text{supp}(M_z) \subset \mathbb{Y}$ only, whereas the function $\tilde{p}_z(\lambda, \nu)$ is always defined on the whole graph \mathbb{Y} . However, the expression for $p_z(\lambda, \nu)$ given in §4.7 makes sense for all couples $\lambda \nearrow \nu$. This makes it possible to say that both functions coincide even for $z \in \mathbb{Z}$.

Proof. a) Let us introduce some notation which will also be used in the sequel. Let ξ_λ , $\lambda \in \mathbb{Y}_n$, be the vector in $H(\lambda) \subset H^n \subset H$ corresponding to χ^λ under the identification $H^n \cong L^2(S(n), \mu_1^n)$. It has unit length and is $K(n)$ -invariant. The distinguished vector of T_z will be denoted as ξ_0 as before. As a vector in $L^2(S(n), \mu_1^n)$ it is given by the function

$$F_z^n(x) = \left(\frac{n!}{t(t+1) \dots (t+n-1)} \right)^{1/2} z^{[x]}; \quad x \in S(n),$$

and can be decomposed as

$$\xi_0 = \sum_{\lambda \in \mathbb{Y}_n} a_z(\lambda) \xi_\lambda,$$

where

$$a_z(\lambda) = \left(\frac{1}{t(t+1) \dots (t+n-1)} \right)^{1/2} \prod_{b \in \lambda} (z + c(b)) \cdot \frac{\dim \lambda}{\sqrt{n!}}.$$

b) We now show that, for every $\nu \in \mathbb{Y}_{n+1}$, the vector ξ_ν can be decomposed as

$$\xi_\nu = \sum_{\lambda \nearrow \nu} \xi_{\lambda\nu},$$

where the vectors $\xi_{\lambda\nu}$ are pairwise orthogonal, $K(n)$ -invariant,

$$(\xi_{\lambda\nu}, \xi_{\lambda\nu}) = \frac{\dim \lambda}{\dim \nu},$$

and $\xi_{\lambda\nu}$ generates, under the action of $G(n)$, the representation $\pi^\lambda \times \pi^\lambda$.

Indeed, let $H(\pi^\nu)$ be the space of an irreducible representation π^ν of the group $S(n+1)$, and let $\text{End } H(\pi^\nu)$ be the algebra of operators on this space. Endow $\text{End } H(\pi^\nu)$ with the inner product

$$(A, B) = \frac{\text{tr}(AB^*)}{\dim \nu}$$

and define an action of the group $G(n+1)$ on $\text{End } H(\pi^\nu)$ by

$$g \cdot A = \pi^\nu(g_1) A \pi^\nu(g_2)^{-1}, \quad A \in \text{End } H(\pi^\nu), \quad g = (g_1, g_2) \in G(n+1).$$

It is convenient to identify the vector spaces $\text{End } H(\pi^\nu)$ and $H(\nu) \subset H^{n+1} = L^2(S(n+1), \mu_1^{n+1})$ as follows: to an operator $A \in \text{End } H(\pi^\nu)$ we assign the function $\hat{A}(s) = \text{tr}(A\pi^\nu(s^{-1}))$. The map $A \mapsto \hat{A}$ preserves the inner product and commutes with the action of the group $G(n+1)$.

Let 1_ν denote the identity operator in the space $H(\pi^\nu)$. Its image under the correspondence $A \mapsto \hat{A}$ coincides with the vector ξ_ν . Further, for any $\lambda \nearrow \nu$, let $1_{\lambda\nu} \in \text{End } H(\pi^\nu)$ denote the orthogonal projection onto the subspace of vectors that transform according to the representation π^λ under the action of $S(n) \subset S(n+1)$. Define $\xi_{\lambda\nu}$ as the image of the operator $1_{\lambda\nu}$ under the correspondence $A \mapsto \hat{A}$. Clearly, the vectors $\xi_{\lambda\nu}$ satisfy all the required properties.

c) Now let us remark that it suffices to prove (5-5-1) when $z \notin \mathbb{Z}$. In fact, the right-hand side of (5-5-1) is continuous in the parameter z ranging over the Riemann sphere. It also follows from the definition (5-3-1) of $\tilde{p}_z(\lambda, \nu)$ that this function is continuous in z , since so is the map L_z^n .

The assumption $z \notin \mathbb{Z}$ implies that $a_z(\lambda) \neq 0$ for all λ , which will be used in the computation below.

d) Equating the decompositions of ξ_0 in ξ_λ , $\lambda \in \mathbb{Y}_n$, and in ξ_ν , $\nu \in \mathbb{Y}_{n+1}$, we conclude that

$$\xi_0 = \sum_{\lambda \in \mathbb{Y}_n} a_z(\lambda) \xi_\lambda = \sum_{\nu \in \mathbb{Y}_{n+1}} a_z(\nu) \xi_\nu.$$

Substituting the decomposition $\xi_\nu = \sum_{\lambda \nearrow \nu} \xi_{\lambda\nu}$, we arrive at

$$\sum_{\lambda \in \mathbb{Y}_n} a_z(\lambda) \xi_\lambda = \sum_{\lambda \in \mathbb{Y}_n} \sum_{\nu \searrow \lambda} a_z(\nu) \xi_{\lambda\nu}.$$

Comparing the components in both sides that transform according to a given irreducible representation of $G(n) \subset G(n+1)$ we see that

$$a_z(\lambda) \xi_\lambda = \sum_{\nu \searrow \lambda} a_z(\nu) \xi_{\lambda\nu} \quad \text{for any } \lambda \in \mathbb{Y}_n.$$

This implies that

$$P_\nu \xi_\lambda = \frac{a_z(\nu)}{a_z(\lambda)} \xi_{\lambda\nu}, \quad \nu \searrow \lambda,$$

whence

$$\tilde{p}_z(\lambda, \nu) = \left| \frac{a_z(\nu)}{a_z(\lambda)} \right|^2 \|\xi_{\lambda\nu}\|^2 = \left| \frac{a_z(\nu)}{a_z(\lambda)} \right|^2 \frac{\dim \lambda}{\dim \nu}$$

by the definition of the transition function. This last equation, along with the explicit formula for the coefficients $a_z(\cdot)$, implies the desired formula for $\tilde{p}_z(\lambda, \nu)$. \square

5.6. The subgraphs $\mathbb{Y}(p, q)$ and levels of Young diagrams. Given a couple (p, q) of nonnegative integers, we define a subset $\mathbb{Y}(p, q) \subset \mathbb{Y}$ as follows

$$\begin{aligned} \mathbb{Y}(p, q) &= \{\lambda \in \mathbb{Y} \mid (p, q) \in \lambda, (p+1, q+1) \notin \lambda\}, & p, q \geq 1 \\ \mathbb{Y}(p, 0) &= \{\lambda \in \mathbb{Y} \mid \lambda_{p+1} = \lambda_{p+2} = \cdots = 0\}, & p \geq 1 \\ \mathbb{Y}(0, q) &= \{\lambda \in \mathbb{Y} \mid (\lambda')_{q+1} = (\lambda')_{q+2} = \cdots = 0\}, & q \geq 1 \\ \mathbb{Y}(0, 0) &= \{\emptyset\} \end{aligned}$$

In other words, the set $\mathbb{Y}(p, q)$, where $p, q \geq 1$, consists of all diagrams containing the rectangle of shape $p \times q$ but not the box $(p+1, q+1)$. The set $\mathbb{Y}(p, 0)$ consists of all diagrams with at most p rows, and the set $\mathbb{Y}(0, q)$ consists of all diagrams with at most q columns.

Each $\mathbb{Y}(p, q)$ may be viewed as a connected subgraph of the Young graph.

Proposition 5.6.1. *Fix an arbitrary integer k and remove from the Young graph all edges $\lambda \nearrow \nu$ such that the content of the box $\nu \setminus \lambda$ equals $-k$. Then we obtain a subgraph in \mathbb{Y} whose connected components are exactly the $\mathbb{Y}(p, q)$'s with $p - q = k$.*

Proof. This follows from the three claims which are readily checked. First, the sets $\mathbb{Y}(p, q)$ with fixed $p - q = k$ form a partition of the set of all Young diagrams. Second, if $\lambda \nearrow \nu$ and $c(\nu \setminus \lambda) \neq k$ then λ and ν belong to one and the same part $\mathbb{Y}(p, q)$ of that partition. Third, if $\lambda \nearrow \nu$ and $c(\nu \setminus \lambda) = k$ then λ and ν belong to different parts: specifically, if $\lambda \in \mathbb{Y}(p, q)$ then $\nu \in \mathbb{Y}(p+1, q+1)$. \square

For an arbitrary $k \in \mathbb{Z}$, we define the k -level of a Young diagram λ as follows

$$\text{lev}_k(\lambda) = \#\{b \in \lambda \mid c(b) = -k\}.$$

In other words, $\text{lev}_0(\lambda)$ equals the length of the main diagonal in λ , and $\text{lev}_k(\lambda)$ is the number of boxes on the diagonal shifted (with respect to the main diagonal) k boxes downwards if $k \geq 0$, and $|k|$ boxes upwards, if $k \leq 0$.

Proposition 5.6.2. *Fix an arbitrary integer k . The partition of the set \mathbb{Y} into disjoint union of the sets $\mathbb{Y}(p, q)$ with $p - q = k$ coincides with the partition according the value of the k -level. Specifically, a given part $\mathbb{Y}(p, q)$ with $p - q = k$ is exactly the set of diagrams with $\text{lev}_k(\cdot) = l$, where $l = \min(p, q)$.*

Proof. This is evident. \square

5.7. The decomposition into blocks.

The knowledge of the transition function $\tilde{p}_z(\lambda, \nu)$, and the results of Propositions 5.3.1 and 5.4.1 lead us to a preliminary decomposition of representations T_z for $z \in \mathbb{Z}$.

Define a function $A_{pq}(\cdot)$ on nonempty diagrams as follows. If $p, q = 1, 2, \dots$ then this is the characteristic function of the set $\mathbb{Y}(p, q)$. If $p = 1, 2, \dots$ and $q = 0$ then this is the characteristic function of $\mathbb{Y}(p, 0) \setminus \{\emptyset\}$. Similarly, if $p = 0$ and $q = 1, 2, \dots$ then this is the characteristic function of $\mathbb{Y}(0, q) \setminus \{\emptyset\}$.

Theorem 5.7.1. *Fix $k \in \mathbb{Z}$ and let (p, q) be a couple of nonnegative integers, not vanishing simultaneously, and such that $p - q = k$.*

(i) *The function $A_{pq}(\cdot)$ satisfies the condition (5-3-1) involving the transition function $\tilde{p}_k(\lambda, \nu)$. Therefore, it determines an operator A_{pq} in the commutant \mathcal{A} of the representation T_k .*

(ii) *Any operator A_{pq} is an orthogonal projection onto a subspace $H_{pq} \subset H$. The subspaces H_{pq} are pairwise orthogonal, and their direct sum is the whole H . Thus, they determine a decomposition of the representation T_k into a direct sum of subrepresentations,*

$$T_k = \bigoplus_{p-q=k} T_{pq}, \quad (p, q) \in \mathbb{Z}_+^2 \setminus \{(0, 0)\}.$$

(iii) Denote by Reg_{pq}^n , where n is large enough, the subrepresentation of the regular representation Reg^n , which is the union of the components $\pi^\lambda \times \pi^\lambda$ such that $\lambda \in \mathbb{Y}(p, q)$. Let H_{pq}^n be the corresponding subspace in H^n . Then the isometric embedding $L_k^n : H^n \rightarrow H^{n+1}$ maps H_{pq}^n into H_{pq}^{n+1} , and the representation $T_{pq} \subset T_k$ coincides with the inductive limit of the representations $\text{Reg}_{pq}^n \subset \text{Reg}^n$ as $n \rightarrow \infty$.

Proof. (i) For $\lambda \in \mathbb{Y}_n \cap \mathbb{Y}(p, q)$ we have a chain of identities

$$A_{pq}(\lambda) = 1 = \sum_{\nu \searrow \lambda} \tilde{p}_k(\lambda, \nu) = \sum_{\nu \searrow \lambda, \nu \in \mathbb{Y}(p, q)} \tilde{p}_k(\lambda, \nu) = \sum_{\nu \searrow \lambda} \tilde{p}_k(\lambda, \nu) A_{pq}(\nu).$$

Therefore, the function $A_{pq}(\cdot)$ satisfies the condition (5-3-1). Since this function is bounded, it determines, according to Proposition 5.3.1, an operator A_{pq} in the commutant \mathcal{A} of the representation T_k .

(ii) From Proposition 5.3.1 and the definition of the function $A_{pq}(\cdot)$ it follows that for all $n = 1, 2, \dots$

$$P_n A_{pq} P_n = \sum_{\lambda \in \mathbb{Y}_n \cap \mathbb{Y}(p, q)} P(\lambda). \quad (5-6-2)$$

Since $P_n A_{pq} P_n$ is an orthoprojector for any n , so is the operator A_{pq} . A similar argument shows that the projectors A_{pq} are pairwise orthogonal and sum up to the identity operator. Note that here we could also use Proposition 5.4.1.

(iii) Since all three operators P_n , A_{pq} , $P_n A_{pq} P_n$ are orthoprojectors, it follows that P_n and A_{pq} commute. Along with (5-6-2) this implies that the operator $P_n A_{pq} = A_{pq} P_n$ projects H^n onto the invariant subspace H_{pq}^n . Since P_{n+1} majorizes P_n , we see that H_{pq}^n is a subspace of H_{pq}^{n+1} . \square

We shall call the subrepresentations $T_{pq} \subset T_k$ the *blocks* of the representation T_k . We shall show below that the blocks are themselves reducible, and find their decomposition into a direct integral of irreducible representations. The only exceptions are the one-dimensional blocks $T_{10} \subset T_1$ and $T_{01} \subset T_{-1}$: each one is generated by the distinguished spherical vector. The block T_{10} is the identity representation, and T_{01} is $\text{sgn} \times \text{sgn}$.

5.8. Another approach to decomposition into blocks. It is worth noting that the decomposition of the representation T_k , $k \in \mathbb{Z}$, into blocks can be obtained in a different way – using the operator A_z intertwining representations T_z and $T_{\bar{z}}$.

Recall that the operator A_z was described in §4.7 as inductive limit, as $n \rightarrow \infty$, of certain operators $A_{n,z} : H^n \rightarrow H^n$ commuting with the representation Reg^n . In other words, A_z preserves H^n for all $n = 1, 2, \dots$, and the restriction of A_z to H^n is $A_{n,z}$. The operator $A_{n,z}$ acts in the space $H^n = L^2(S(n), \mu_1^n)$ as the operator of convolution with the central function

$$\Theta_{n,z} = \sum_{\lambda \vdash n} \left(\prod_{b \in \lambda} \frac{\bar{z} + c(b)}{z + c(b)} \right) \dim \lambda \cdot \chi^\lambda$$

on the group $S(n)$ (see Proposition 4.7.1).

This function is correctly defined for all nonintegral values of the parameter z . If $z \in \mathbb{R} \setminus \mathbb{Z}$, then the function $\Theta_{n,z}$ is just $\delta_e = \sum_{\lambda \vdash n} \dim \lambda \cdot \chi^\lambda$, the delta function

at the identity element of the group $S(n)$. Therefore, the associated operator is simply the identity operator in H^n . Let us see what happens to $\Theta_{n,z}$ when the parameter $z \in \mathbb{C} \setminus \mathbb{R}$ approaches an integer point $k \in \mathbb{Z}$. It turns out that there exists a nontrivial limit of $\Theta_{n,z}$, depending this time on the direction in which z approaches k .

In order to see this, set $z = k + \varepsilon w$, where $w \neq 0$ is a fixed complex number and ε is a real number going to 0. Set $u = \bar{w}/w$. Then one can easily see that

$$\lim_{\varepsilon \rightarrow 0} \Theta_{n,z} = \sum_{\lambda \vdash n} u^{\text{lev}_k(\lambda)} \dim \lambda \cdot \chi^\lambda.$$

Denote by $A_{n,k}(u)$ the convolution operator with the latter function. Clearly, for each fixed u , $|u| = 1$, the sequence of operators $(A_{n,k}(u))$, $n = 1, 2, \dots$, is consistent with the embeddings $L_k^n : H^n \rightarrow H^{n+1}$ and hence determines an operator $A_k(u)$ in the space H of the representation T_k , commuting with this representation.

On the other hand,

$$A_k(u)|_{H_{pq}^n} = u^l \cdot 1, \quad l = \min(p, q),$$

which implies that

$$A_k(u) = \sum_{p-q=k} u^l A_{pq},$$

i.e., $A_k(u)$ can be considered as a generating function of the projectors A_{pq} . One can derive from this fact another proof of Theorem 5.6.1, not relying on Proposition 5.3.1.

6. THE INVARIANT VECTORS

Let us outline the contents of the present Section.

We start with arbitrary z and describe a convenient realization of the space $V_z \subset H(T_z)$ formed by K -invariant vectors. Specifically, we show that V_z is isomorphic to the space \mathcal{F}_z of functions on \mathbb{Y} that satisfying two conditions: a harmonicity type condition and a Hardy type condition. In these terms, if z is an integer, the splitting of the space V_z induced by the block decomposition $T_z = \bigoplus_{p-q=z} T_{pq}$ takes especially nice form.

Then we focus on the case when z is an integer. We prove two main results: Theorems 6.2.1 and 6.2.2. In Theorem 6.2.1 we construct, for any block T_{pq} , a certain K -invariant vector v_{pq} . In Theorem 6.2.2 we compute the spectral decomposition of the corresponding spherical function: we show that the spectral measure lives on a finite dimensional face $\Omega(p, q)$ of the simplex Ω .

Later on in §7 we shall show that, for any couple (p, q) , the vector v_{pq} is a cyclic vector in T_{pq} . This allows us to completely understand the spectral decomposition of T_z at the integer points z .

6.1. The space \mathcal{F}_z . Recall that the vectors ξ_λ were introduced in §5.5 (see the proof of Theorem 5.5.1, part a)).

Proposition 6.1.1. *For any $n = 1, 2, \dots$ and any $\lambda \in \mathbb{Y}_n$, $\nu \in \mathbb{Y}_{n+1}$,*

$$(\xi_\lambda, \xi_\nu) = \begin{cases} (z + c_{\lambda\nu}) / \sqrt{(t+n)(n+1)}, & \text{if } \lambda \nearrow \nu \\ 0, & \text{otherwise,} \end{cases}$$

where, as before, $c_{\lambda\nu}$ is the content of the box ν/λ , $t = |z|^2$.

Proof. The inner product (ξ_λ, ξ_ν) is a continuous function in z , hence it suffices to prove the formula under the assumption $z \notin \mathbb{Z}$. In this case $(\xi_0, \xi_\lambda) \neq 0$ for all λ . Indeed, this follows from the equality $(\xi_0, \xi_\lambda) = a_z(\lambda)$ and the explicit expression for $a_z(\lambda)$, see the proof of Theorem 5.5.1, part a).

Denote by Q_λ the projection operator from the Hilbert space H of the representation T_z to the subspace of all vectors that transform, under the action of the subgroup $G(n)$, according to the representation $\pi^\lambda \otimes \pi^\lambda$ (that is, the range of Q_λ is the isotypical component of $\pi^\lambda \otimes \pi^\lambda$ in $T_z|_{G(n)}$). Notice that $P(\lambda) \leq Q_\lambda$ and $Q_\lambda \xi_\lambda = \xi_\lambda$. It follows from the proof of Theorem 5.5.1, part b), that

$$(Q_\lambda \xi_\nu, Q_\lambda \xi_\nu) = \begin{cases} \dim \lambda / \dim \nu, & \text{if } \lambda \nearrow \nu \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if the condition $\lambda \nearrow \nu$ does not hold then $Q_\lambda \xi_\nu = 0$ and $(\xi_\lambda, \xi_\nu) = 0$.

Consider the decompositions

$$\xi_0 = \sum_{\lambda \in \mathbb{Y}_n} (\xi_0, \xi_\lambda) \xi_\lambda = \sum_{\nu \in \mathbb{Y}_{n+1}} (\xi_0, \xi_\nu) \xi_\nu.$$

Applying the operator Q_λ we derive

$$Q_\lambda \xi_0 = (\xi_0, \xi_\lambda) \xi_\lambda = \sum_{\nu \searrow \lambda} (\xi_0, \xi_\nu) Q_\lambda \xi_\nu.$$

Fix a diagram ν such that $\nu \searrow \lambda$. Taking the inner product with ξ_ν we obtain

$$(\xi_0, \xi_\lambda) (\xi_\lambda, \xi_\nu) = (\xi_0, \xi_\nu) (Q_\lambda \xi_\nu, \xi_\nu).$$

We have used here the relation $(Q_\lambda \xi_{\nu'}, \xi_\nu) = 0$ for any $\nu' \in \mathbb{Y}_{n+1} \setminus \{\nu\}$, which in turn follows from the fact that $Q_\lambda H(\nu) \subset H(\nu)$ for any $\nu \in \mathbb{Y}_{n+1}$.

Now remark that

$$(Q_\lambda \xi_\nu, \xi_\nu) = (Q_\lambda \xi_\nu, Q_\lambda \xi_\nu) = \frac{\dim \lambda}{\dim \nu},$$

and hence

$$(\xi_0, \xi_\lambda) (\xi_\lambda, \xi_\nu) = (\xi_0, \xi_\nu) \frac{\dim \lambda}{\dim \nu}.$$

Since $(\xi_0, \xi_\lambda) \neq 0$ this implies

$$(\xi_\lambda, \xi_\nu) = \frac{(\xi_0, \xi_\nu)}{(\xi_0, \xi_\lambda)} \frac{\dim \lambda}{\dim \nu} = \frac{a_z(\nu)}{a_z(\lambda)} \frac{\dim \lambda}{\dim \nu}.$$

Substituting the explicit expression for $a_z(\cdot)$ (see the proof of Theorem 5.5.1 part a)) concludes the proof. \square

Definition 6.1.2. Denote by \mathcal{F}_z the space of complex-valued functions $f(\lambda)$ on the vertices $\lambda \neq \emptyset$ of the Young graph, satisfying the following two conditions.

(i) *Pseudoharmonicity*: for any $\lambda \in \mathbb{Y}_n$, $n = 1, 2, \dots$,

$$f(\lambda) = \sum_{\nu \searrow \lambda} f(\nu) (\xi_\nu, \xi_\lambda) = \sum_{\nu \searrow \lambda} f(\nu) \frac{\bar{z} + c_{\lambda\nu}}{\sqrt{(t+n)(n+1)}}$$

(we have used here the formula of Proposition 6.1.1).⁸

(ii) *Hardy type condition*: for any $n = 1, 2, \dots$

$$\|f\|^2 := \sup_n \sum_{\lambda \in \mathbb{Y}_n} |f(\lambda)|^2 < \infty.$$

It is worth noting that for any f satisfying the pseudoharmonicity condition, the sum $\sum_{\lambda \in \mathbb{Y}_n} |f(\lambda)|^2$ does not decrease, as $n \rightarrow \infty$ (this follows from the proof of Proposition 6.1.3 below). This shows that we could equally well define the norm $\|f\|$ by the formula

$$\|f\|^2 = \lim_{n \rightarrow \infty} \sum_{\lambda \in \mathbb{Y}_n} |f(\lambda)|^2.$$

Proposition 6.1.3. *Let V_z be the subspace of K -invariant vectors of the representation T_z . Then the map*

$$v \mapsto f_v, \quad f_v(\lambda) = (v, \xi_\lambda)$$

provides a linear isomorphism $V_z \rightarrow \mathcal{F}_z$, preserving the norm.

Proof. Recall that by P_n we denote the orthogonal projection from $H = H(T_z)$ onto H^n . Denote by V^n the subspace of $K(n)$ -invariant vectors in H^n . Note that V^n is not contained in V_z , but $P_n V_z \subseteq V^n$ and $P_n V^{n+1} \subseteq V^n$. The vectors ξ_λ , where $\lambda \in \mathbb{Y}_n$, form an orthonormal basis in V^n . Given two vectors

$$v_n = \sum_{\lambda \in \mathbb{Y}_n} a(\lambda) \xi_\lambda \in V^n \quad \text{and} \quad v_{n+1} = \sum_{\nu \in \mathbb{Y}_{n+1}} b(\nu) \xi_\nu \in V^{n+1},$$

we have

$$v_n = P_n v_{n+1} \iff a(\lambda) = \sum_{\nu \searrow \lambda} b(\nu) (\xi_\nu, \xi_\lambda) \quad \forall \lambda \in \mathbb{Y}_n.$$

Note that these relations imply that

$$\sum_{\lambda \in \mathbb{Y}_n} |a(\lambda)|^2 = \|v_n\|^2 \leq \|v_{n+1}\|^2 = \sum_{\nu \in \mathbb{Y}_{n+1}} |b(\nu)|^2.$$

Assume now that $v \in V_z$ and $v_n = P_n v$ for $n = 1, 2, \dots$. Then

$$f_v(\lambda) = (v, \xi_\lambda) = (v_n, \xi_\lambda), \quad n = |\lambda|.$$

⁸Cf. the definition of harmonic functions on \mathbb{Y} , see the end of §9.3

The above argument shows that the function f_v satisfies the pseudoharmonicity condition and

$$\|f_v\|^2 := \sup_n \|v_n\|^2 = \lim_{n \rightarrow \infty} \|v_n\|^2 = \|v\|^2 < \infty,$$

hence $f_v \in \mathcal{F}_z$. Therefore, the function $v \mapsto f_v$ provides an isometric embedding $V_z \rightarrow \mathcal{F}_z$.

Let us show now that the map is surjective. Given an arbitrary $f \in \mathcal{F}_z$, we set

$$v_n = \sum_{\lambda \in \mathbb{Y}_n} f(\lambda) \xi_\lambda, \quad n = 1, 2, \dots$$

Then $v_n \in V^n$, $v_n = P_n v_{n+1}$ and

$$\lim_{n \rightarrow \infty} \|v_n\|^2 = \sup_n \|v_n\|^2 = \|f\|^2.$$

Therefore, the vectors v_n converge to a vector $v \in H(T_z)$. For every m the vector v is $K(m)$ -invariant, because the vectors v_n possess this property for all $n \geq m$. Hence, $v \in V_z$ and it follows that $f = f_v$. \square

6.2. Two theorems. We assume from this point to the end of §6 that z is an integer and we write $z = k$. Recall that in §5 we have introduced invariant subspaces $H(T_{pq}) \subset H(T_k)$ (with the indices p, q subject to the condition $p - q = k$), called the blocks. Denote by $V_{pq} = V \cap H(T_{pq})$ the subspace of all K -invariant vectors in the block $H(T_{pq})$. Let \mathcal{F}_{pq} be the subspace in \mathcal{F}_k formed by the functions supported by $\mathbb{Y}(p, q) \subset \mathbb{Y}$. The decomposition

$$\mathcal{F}_k = \bigoplus_{p-q=k} \mathcal{F}_{pq}$$

is parallel to

$$V_k = \bigoplus_{p-q=k} V_{pq},$$

and the latter corresponds to the decomposition

$$H(T_k) = \bigoplus_{p-q=k} H(T_{pq}),$$

of §5.

Our goal is to produce a certain vector $v_{pq} \in V_{pq}$, which will be described in terms of the realization $V_{pq} = \mathcal{F}_{pq}$. The following encoding of a Young diagram $\lambda \in \mathbb{Y}(p, q)$ will be convenient. Recall that λ belongs to $\mathbb{Y}(p, q)$ if and only if λ contains the rectangle of shape $p \times q$ (p rows and q columns) but not the box $(p+1, q+1)$. One can represent such a diagram as union of three parts: the $p \times q$ -rectangle, a diagram λ^+ to its right, and a diagram below the rectangle. The transpose of the latter diagram will be denoted by λ^- . Formally:

$$\lambda^+ = (\lambda_1^+, \dots, \lambda_p^+), \quad \text{where } \lambda_i^+ = \lambda_i - q, \quad 1 \leq i \leq p;$$

$$\lambda^- = (\lambda_1^-, \dots, \lambda_q^-), \quad \text{where } \lambda_j^- = (\lambda')_j - p, \quad 1 \leq j \leq q.$$

If $q = 0$ then $\lambda^- = \emptyset$ and $\lambda = \lambda^+$. Likewise, if $p = 0$ then $\lambda^+ = \emptyset$ and $\lambda = (\lambda^-)'$.

If both p, q are nonzero then the correspondence $\lambda \mapsto (\lambda^+, \lambda^-)$ establishes a bijection between the sets $\mathbb{Y}(p, q)$ and $\mathbb{Y}(p, 0) \times \mathbb{Y}(q, 0)$.

Below we set $n = |\lambda|$.

Theorem 6.2.1. *Let $z = k$ be an integer and let T_{pq} be an arbitrary block of the representation T_k (recall that p, q are nonnegative integers such that $p - q = k$ and $(p, q) \neq (0, 0)$ if $k = 0$).*

In the space $H(T_{pq})$ there is a K -invariant vector v_{pq} , such that the corresponding function $f_{pq}(\lambda) = f_{v_{pq}}(\lambda)$ in the space $\mathcal{F}_{pq} \subset \mathcal{F}_z$ of functions on the graph $\mathbb{Y}(p, q)$ has the following form

$$f_{pq}(\lambda) = (-1)^{|\lambda^-|} \frac{\sqrt{((p-q)^2 + n - 1)! n!}}{(p^2 + q^2 - pq + n - 1)!} \\ \times \prod_{1 \leq i < j \leq p} (\lambda_i^+ - \lambda_j^+ + j - i) \prod_{1 \leq r < s \leq q} (\lambda_r^- - \lambda_s^- + s - r),$$

where $n = |\lambda|$.

In the next theorem we use the concept of *spectral measure*; it is explained in §9.7. We also need the finite-dimensional faces $\Omega(p, q) \subset \Omega$, which are defined in §9.6 ($\Omega(p, q)$ is a simplex of dimension $p + q - 1$).

Theorem 6.2.2. *Let v_{pq} be the vector defined in Theorem 6.2.1, let*

$$\varphi_{pq}(g) = \frac{(T_z(g)v_{pq}, v_{pq})}{\|v_{pq}\|^2} = \frac{(T_{pq}(g)v_{pq}, v_{pq})}{\|v_{pq}\|^2}, \quad g \in G,$$

be the corresponding spherical function on the group G , and let σ_{pq} be the spectral measure on Ω , corresponding to φ_{pq} .

The measure σ_{pq} is supported by $\Omega(p, q) \subset \Omega$ and has the density

$$\|v_{pq}\|^{-2} \prod_{1 \leq i < j \leq p} (\alpha_i - \alpha_j)^2 \prod_{1 \leq r < s \leq q} (\beta_r - \beta_s)^2$$

with respect to Lebesgue measure on $\Omega(p, q)$.

In order to prove these two theorems, we need a few lemmas.

6.3. Preparatory Lemmas.

Lemma 6.3.1. *Let $p = 1, 2, \dots$ be fixed. The function $g_p(\lambda)$ on the graph $\mathbb{Y}(p, 0)$, determined by*

$$g_p(\lambda) = \prod_{1 \leq i < j \leq p} (\lambda_i - \lambda_j + j - i),$$

satisfies the relation

$$(p^2 + |\lambda|) g_p(\lambda) = \sum_{\nu \searrow \lambda} (p + c_{\lambda\nu}) g_p(\nu).$$

Proof. By setting

$$l_i = \lambda_i + p - i \quad (i = 1, \dots, p), \quad |l| = l_1 + \dots + l_p, \\ V(l_1, \dots, l_p) = \prod_{1 \leq i < j \leq p} (l_i - l_j),$$

we transform the relation to the form

$$\left(p^2 - \frac{p(p-1)}{2} + |l|\right) V(l_1, \dots, l_p) = \sum_{i=1}^p (l_i + 1) V(l_1, \dots, l_i + 1, \dots, l_p).$$

The sum extends over all indices $i = 1, \dots, p$, not just those with $l_{i-1} > l_i + 1$, because in case of $l_{i-1} = l_i + 1$ the term $V(l_1, \dots, l_i + 1, \dots, l_p)$ vanishes.

Let us check the latter relation. The right hand side, being a skew symmetric polynomial in l_1, \dots, l_p , is divisible by $V(l_1, \dots, l_p)$. Since its highest term is $|l| V(l_1, \dots, l_p)$, the right hand side has the form

$$(|l| + \text{const}) V(l_1, \dots, l_p).$$

In order to determine the constant, we specialize the identity

$$\sum_{i=1}^p (l_i + 1) V(l_1, \dots, l_i + 1, \dots, l_p) = (|l| + \text{const}) V(l_1, \dots, l_p).$$

at

$$(l_1, l_2, \dots, l_p) = (p-1, p-2, \dots, 0),$$

Then we obtain

$$p V(p, p-2, p-3, \dots, 0) = \left(\frac{p(p-1)}{2} + \text{const}\right) V(p-1, p-2, \dots, 0),$$

whence

$$\text{const} = p^2 - \frac{p(p-1)}{2} = \frac{p(p+1)}{2},$$

and we are done. \square

Remark 6.3.2. One can suggest another proof of Lemma 6.3.1. It is seemingly more round about, but better explains the origin of the result. The idea is that in case of $p = k$, $q = 0$ we already dispose of an example of a pseudoharmonic function on the Young lattice $\mathbb{Y}(p, 0)$: the function

$$\tilde{g}_p(\lambda) := (\xi_0, \xi_\lambda) = \left(\frac{1}{(p^2)_n}\right)^{1/2} \frac{\dim \lambda}{\sqrt{n!}} \prod_{b \in \lambda} (p + c(b)), \quad n = |\lambda|$$

(cf. the proof of Theorem 5.5.1, part a); in this setup $t = p^2$). Since

$$\frac{\dim \lambda}{n!} = \frac{V(l_1, \dots, l_p)}{l_1! \dots l_p!}$$

and

$$\prod_{b \in \lambda} (p + c(b)) = \frac{l_1! \dots l_p!}{(p-1)! \dots 0!},$$

we obtain that

$$\tilde{g}_p(\lambda) = \frac{1}{(p-1)! \dots 0!} \left(\frac{n!}{(p^2)_n}\right)^{1/2} g_p(\lambda),$$

and the required relation for $g_p(\lambda)$ follows from the pseudoharmonicity condition for $\tilde{g}_p(\lambda)$. \square

Using Lemma 6.3.1 and the fact that the graph $\mathbb{Y}(p, q)$ is isomorphic to the product $\mathbb{Y}(p, 0) \times \mathbb{Y}(q, 0)$ we will produce a pseudoharmonic function on $\mathbb{Y}(p, q)$.

Let $(x_n)_{n \geq pq}$ be a sequence of positive real numbers satisfying the recurrence relation

$$\frac{x_{n+1}}{x_n} = \frac{n + p^2 + q^2 - pq}{\sqrt{(n + (p - q)^2)(n + 1)}}.$$

For instance, one can set

$$x_n = \frac{\Gamma(p^2 + q^2 - pq + n)}{\sqrt{\Gamma((p - q)^2 + n)\Gamma(n + 1)}}. \quad (6-3-1)$$

Below we use the correspondence $\lambda \mapsto (\lambda^+, \lambda^-)$ introduced in §6.2.

Lemma 6.3.3. *Let p, q be nonnegative integers, $(p, q) \neq (0, 0)$. The function $f_{pq}(\lambda)$ on the graph $\mathbb{Y}(p, q)$ determined by the formula*

$$f_{pq}(\lambda) = \frac{(-1)^{|\lambda^-|}}{x_n} \prod_{1 \leq i < j \leq p} (\lambda_i^+ - \lambda_j^+ + j - i) \prod_{1 \leq r < s \leq p} (\lambda_r^- - \lambda_s^- + s - r),$$

satisfies the pseudoharmonicity condition with the parameter $k = p - q$, i.e.,

$$f_{pq}(\lambda) = \sum_{\nu \searrow \lambda} \frac{p - q + c_{\lambda\nu}}{\sqrt{((p - q)^2 + n)(n + 1)}} f_{pq}(\nu)$$

for any $\lambda \in \mathbb{Y}(p, q) \cap \mathbb{Y}_n$.

In this formula we assume that ν belongs to $\mathbb{Y}(p, q)$. However, the formula remains true without this assumption, because if $\lambda \nearrow \nu$, $\lambda \in \mathbb{Y}(p, q)$, and $\nu \notin \mathbb{Y}(p, q)$ then the factor $p - q + c_{\lambda\nu}$ vanishes.

Proof. Assume first that both p and q are positive. In the notation of Lemma 6.3.1,

$$f_{pq}(\lambda) = \frac{(-1)^{|\lambda^-|}}{x_n} g_p(\lambda^+) g_q(\lambda^-).$$

Let $\nu \in \mathbb{Y}(p, q)$ denote a Young diagram with $n + 1$ boxes. The condition $\lambda \nearrow \nu$ implies one of the following two conditions:

- (i) $\lambda^+ \nearrow \nu^+$, $\lambda^- = \nu^-$;
- (ii) $\lambda^+ = \nu^+$, $\lambda^- \nearrow \nu^-$.

Note that

$$c_{\lambda\nu} = \begin{cases} q + c_{\lambda^+\nu^+}, & \text{in case (i),} \\ -p - c_{\lambda^-\nu^-}, & \text{in case (ii).} \end{cases}$$

Therefore,

$$p - q + c_{\lambda\nu} = \begin{cases} p + c_{\lambda^+\nu^+}, & \text{in case (i),} \\ -(q + c_{\lambda^-\nu^-}), & \text{in case (ii).} \end{cases}$$

It follows that

$$\begin{aligned}
& \sum_{\nu \searrow \lambda} (p - q + c_{\lambda\nu}) f_{pq}(\nu) = \\
&= \sum_{\nu^+ \searrow \lambda^+} \frac{(-1)^{|\nu^-|}}{x_{n+1}} (p + c_{\lambda^+ \nu^+}) g_p(\nu^+) g_q(\lambda^-) \\
&+ \sum_{\nu^- \searrow \lambda^-} \frac{(-1)^{|\nu^-|+1}}{x_{n+1}} (q + c_{\lambda^- \nu^-}) g_p(\lambda^+) g_q(\nu^-).
\end{aligned}$$

In the first sum $|\nu^-| = |\lambda^-|$, and in the second one $|\nu^-| = |\lambda^-| + 1$. Hence, the sign in both formulas is $(-1)^{|\lambda^-|}$.

Applying the Lemma 6.3.1, we derive that the last expression equals

$$\begin{aligned}
& \frac{(-1)^{|\lambda^-|}}{x_{n+1}} (p^2 + |\lambda^+| + q^2 + |\lambda^-|) g_p(\lambda^+) g_q(\lambda^-) = \\
&= \frac{x_n}{x_{n+1}} (p^2 + q^2 + n - pq) f_{pq}(\lambda).
\end{aligned}$$

We have shown that

$$\sum_{\nu \searrow \lambda} (p - q + c_{\lambda\nu}) f_{pq}(\nu) = \frac{x_n}{x_{n+1}} (p^2 + q^2 + n - pq) f_{pq}(\lambda).$$

Dividing both sides by $\sqrt{(n + (p - q)^2)(n + 1)}$ we obtain that the coefficient of $f_{pq}(\lambda)$ in the right-hand side equals

$$\frac{x_n}{x_{n+1}} \frac{p^2 + q^2 + n - pq}{\sqrt{(n + (p - q)^2)(n + 1)}},$$

which is 1 by definition of the numbers x_n . The lemma follows.

If $q = 0$ or $p = 0$, our relation is simply equivalent to Lemma 6.3.1. \square

Lemma 6.3.4. *The function $f_{pq}(\lambda)$ introduced in Lemma 6.3.3 satisfies the Hardy type condition of Definition 6.1.2. Hence, $f_{pq} \in \mathcal{F}_{pq}$.*

Proof. In order to simplify the notation, set $a = \lambda^+$, $b = \lambda^-$, i.e.,

$$a_1 = \lambda_1^+, \dots, a_p = \lambda_p^+; \quad b_1 = \lambda_1^-, \dots, b_q = \lambda_q^-.$$

When λ ranges over $Y(p, q) \cap \mathbb{Y}_n$, the couple (a, b) ranges over the subset in \mathbb{Z}_+^{p+q} determined by the conditions

$$a_1 \geq \dots \geq a_p, \quad b_1 \geq \dots \geq b_q, \quad \sum a_i + \sum b_j = n - pq.$$

We now have

$$\sum_{\substack{\lambda \in \mathbb{Y}(p, q) \\ |\lambda| = n}} |f_{pq}(\lambda)|^2 = \frac{1}{x_n^2} \sum_{a, b} \prod_{1 \leq i < j \leq p} (a_i - a_j + j - i)^2 \prod_{1 \leq r < s \leq q} (b_r - b_s + s - r)^2.$$

Assuming here $n > pq$, we set

$$\bar{\iota}_n(\lambda) = (\alpha; \beta) = (\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q),$$

where

$$\alpha_1 = \frac{a_1}{n - pq}, \dots, \alpha_p = \frac{a_p}{n - pq}; \quad \beta_1 = \frac{b_1}{n - pq}, \dots, \beta_q = \frac{b_q}{n - pq}.$$

The vector $(\alpha; \beta)$ belongs to the lattice $\frac{1}{n-pq} \mathbb{Z}_+^{p+q}$ and satisfies the conditions

$$\alpha_1 \geq \dots \geq \alpha_p, \quad \beta_1 \geq \dots \geq \beta_q, \quad \sum \alpha_i + \sum \beta_j = 1.$$

In terms of $(\alpha; \beta)$, the expression for $\sum |f_{pq}(\lambda)|^2$ can be written as

$$\frac{n^{p(p-1)+q(q-1)}}{x_n^2} \sum_{(\alpha; \beta)} \prod_{1 \leq i < j \leq p} (\alpha_i - \alpha_j)^2 \prod_{1 \leq r < s \leq q} (\beta_r - \beta_s)^2 \left(1 + O\left(\frac{1}{n}\right)\right).$$

summed over all $(\alpha; \beta)$ subject to the conditions stated above, that is, over the set

$$\Omega(p, q) \cap \left(\frac{1}{n - pq} \mathbb{Z}^{p+q}\right),$$

where $\Omega(p, q)$ is the simplex introduced in §9.6.

From the well-known formula

$$\frac{\Gamma(n + \text{const})}{\Gamma(n)} \sim n^{\text{const}}, \quad n \rightarrow \infty,$$

we get

$$x_n \sim n^{p^2+q^2-pq-\frac{(p-q)^2+1}{2}} = n^{\frac{p^2+q^2-1}{2}},$$

hence the coefficient in front of the sum has the asymptotics (as $n \rightarrow \infty$)

$$n^{p(p-1)+q(q-1)-(p^2+q^2-1)} = n^{-(p+q-1)}.$$

But $p + q - 1$ is exactly the dimension of $\Omega(p, q)$, hence we have

$$\lim_{n \rightarrow \infty} \sum_{\lambda \in \mathbb{Y}(p, q) \cap \mathbb{Y}_n} |f_{pq}(\lambda)|^2 = \int_{\omega = (\alpha; \beta) \in \Omega(p, q)} \prod_{1 \leq i < j \leq p} (\alpha_i - \alpha_j)^2 \prod_{1 \leq r < s \leq q} (\beta_r - \beta_s)^2 d\omega,$$

where $d\omega$ is Lebesgue measure on $\Omega(p, q)$. Since the integral in the right-hand side is finite, we conclude that the Hardy type condition is satisfied. \square

6.4. Proof of Theorems 6.2.1 and 6.2.2. The definition of the function $f_{pq}(\lambda)$ in Theorem 6.2.1 is identical with that given in Lemma 6.3.3. By virtue of Lemmas 6.3.3 and 6.3.4, the function f_{pq} belongs to \mathcal{F}_{pq} . Therefore, it determines a K -invariant vector $v_{pq} \in H(T_{pq})$, which concludes the proof of the theorem.

Let us turn to Theorem 6.2.2. Compare two probability measures on \mathbb{Y}_n .

The first measure, denoted as $M_{pq}^{(n)}$, comes from the coherent system corresponding to the spherical function φ_{pq} introduced in the statement of Theorem 6.2.2,

$$M_{pq}^{(n)}(\lambda) = \frac{\|Q(\lambda) v_{pq}\|^2}{\|v_{pq}\|^2}, \quad \lambda \in \mathbb{Y}_n.$$

By the general theory (see Theorem 9.7.3), the spectral measure σ_{pq} is the weak limit of the measures $\iota_n(M_{pq}^{(n)})$, where $\iota_n : \mathbb{Y}_n \rightarrow \Omega$ is the embedding defined just before Theorem 9.7.3.

The second measure, which we denote as $\overline{M}_{pq}^{(n)}$, has the form

$$\overline{M}_{pq}^{(n)}(\lambda) = \frac{\|Q_\lambda v_{pq}^{(n)}\|^2}{\|v_{pq}^{(n)}\|^2} = \frac{|f_{pq}(\lambda)|^2}{\|v_{pq}^{(n)}\|^2}, \quad \lambda \in \mathbb{Y}_n,$$

where $v_{pq}^{(n)} = P_n v_{pq}$ is the projection of v_{pq} onto the subspace H^n , and the projection Q_λ was defined in §6.1. We know that $\overline{M}_{pq}^{(n)}$ is concentrated on the subset $\mathbb{Y}_n(p, q) = \mathbb{Y}(p, q) \cap \mathbb{Y}_n$.

Below we use the standard norm on signed Borel measures: given such a measure μ , its norm $\|\mu\|$ is defined as the variance of μ_+ plus the variance of μ_- , where $\mu = \mu_+ - \mu_-$ stands for the Jordan decomposition of μ .

Lemma 6.4.1. *We have*

$$\|M_{pq}^{(n)} - \overline{M}_{pq}^{(n)}\| \rightarrow 0, \quad n \rightarrow \infty.$$

Proof. Indeed, as is well known, for any two *probability* Borel measures μ_1, μ_2 , defined on one and the same Borel space,

$$\|\mu_1 - \mu_2\| \leq 2 \sup_X |\mu_1(X) - \mu_2(X)|,$$

where the supremum is taken over arbitrary Borel subsets. Let us apply this inequality to $\mu_1 = M_{pq}^{(n)}$, $\mu_2 = \overline{M}_{pq}^{(n)}$. To simplify the notation, set

$$\xi_1 = \|v_{pq}\|^{-1} v_{pq}, \quad \xi_2 = \|v_{pq}^{(n)}\|^{-1} v_{pq}^{(n)}$$

and, for any subset $X \subset \mathbb{Y}_n$, set

$$Q_X = \sum_{\lambda \in X} Q_\lambda.$$

The operators Q_λ , with λ ranging over \mathbb{Y}_n , are pairwise orthogonal projectors whose sum equal 1, whence $\|Q_X\| \leq 1$ for any X . It follows

$$\|M_{pq}^{(n)} - \overline{M}_{pq}^{(n)}\| \leq 2 \sup_{X \subset \mathbb{Y}_n} |(Q_X \xi_1, \xi_1) - (Q_X \xi_2, \xi_2)| \leq 4 \|\xi_1 - \xi_2\|.$$

But $\|\xi_1 - \xi_2\| \rightarrow 0$ as $n \rightarrow \infty$, because $\|v_{pq} - v_{pq}^{(n)}\| \rightarrow 0$. This completes the argument. \square

On the other hand, it follows from the proof of Lemma 6.3.4 that the measures $\bar{\iota}_n(\bar{M}_{pq}^{(n)})$ on $\Omega(p, q)$ weakly converge to a probability measure, which is absolutely continuous with respect to Lebesgue measure on $\Omega(p, q)$ and whose density is exactly as required in the statement of the theorem.

Lemma 6.4.2. *The measures $\iota_n(\bar{M}_{pq}^{(n)})$ on Ω weakly converge, as $n \rightarrow \infty$, to the same limit measure concentrated on $\Omega(p, q)$.*

Proof. Indeed, let us compare the two embeddings,

$$\bar{\iota}_n : \mathbb{Y}_n(p, q) \rightarrow \Omega(p, q) \subset \Omega \quad \text{and} \quad \iota_n : \mathbb{Y}_n(p, q) \rightarrow \Omega.$$

For any $\lambda \in \mathbb{Y}_n(p, q)$, we can write (viewing $\bar{\iota}_n(\lambda)$ as an element of Ω)

$$\bar{\iota}_n(\lambda) = (\bar{\alpha}_1, \bar{\alpha}_2, \dots; \bar{\beta}_1, \bar{\beta}_2, \dots),$$

where

$$\bar{\alpha}_{p+1} = \bar{\alpha}_{p+2} = \dots = \bar{\beta}_{q+1} = \bar{\beta}_{q+2} = \dots = 0,$$

and similarly

$$\iota_n(\lambda) = (\alpha_1, \alpha_2, \dots; \beta_1, \beta_2, \dots),$$

where

$$\alpha_{d+1} = \alpha_{d+2} = \dots = \beta_{d+1} = \beta_{d+2} = \dots = 0, \quad d = \max(p, q).$$

Further, from the definition of $\bar{\iota}_n(\lambda)$ and $\iota_n(\lambda)$ it follows that for any fixed i, j ,

$$|\bar{\alpha}_i - \alpha_i| \leq \frac{\text{const}}{n}, \quad |\bar{\beta}_j - \beta_j| \leq \frac{\text{const}}{n}$$

where the constant does not depend on λ . Consequently, if F is an arbitrary bounded continuous function on $\mathbb{R}^\infty \times \mathbb{R}^\infty$ depending on finitely many coordinates only, then the result of coupling between f and $\bar{\iota}_n(\bar{M}_{pq}^{(n)}) - \iota_n(\bar{M}_{pq}^{(n)})$ is $O(1/n)$. Since Ω is a compact subset of $\mathbb{R}^\infty \times \mathbb{R}^\infty$, the same is true for any continuous test function F on Ω , which completes the proof. \square

By Lemma 6.4.1,

$$\|\bar{\iota}_n(M_{pq}^{(n)}) - \bar{\iota}_n(\bar{M}_{pq}^{(n)})\| \rightarrow 0, \quad n \rightarrow \infty,$$

so that the measures $\iota_n(M_{pq}^{(n)})$ must have the same limit as the measures $\iota_n(\bar{M}_{pq}^{(n)})$. Together with Lemma 6.4.2 this completes the proof of Theorem 6.2.2.

7. THE COMMUTANT OF REPRESENTATION T_{pq}

7.1. The results. Recall that if z is an integer, $z = k \in \mathbb{Z}$, then the representation $T_z = T_k$ splits into a direct sum of subrepresentations called the blocks,

$$T_k \sim \bigoplus_{p-q=k} T_{pq}$$

where p, q are assumed to be nonnegative integers, and $(p, q) \neq (0, 0)$ if $k = 0$ (see §5.7).

The main goal of this Section is to prove the following result.

Theorem 7.1.1. *Let $z = k$ be an integer and let T_{pq} be an arbitrary block of the representation T_k . Then the vector $v_{pq} \in H(T_{pq})$ constructed in §6 is a cyclic vector of T_{pq} .*

Along with the Theorem 6.2.2 this implies our main result on representations T_k at integer points $z = k$:

Theorem 7.1.2. *The block $T_{pq} \subset T_z$, where $z = k$ is an integer and $p - q = k$, is equivalent to the direct integral of irreducible spherical representations labelled by the points ω of the finite dimensional face $\Omega(p, q)$ of the Thoma simplex, with respect to Lebesgue measure $d\omega$.*

Corollary 7.1.3. *The representations T_{pq} , where $p, q \in \mathbb{Z}$, are pairwise disjoint. In particular, the representations T_k , $k \in \mathbb{Z}$, are pairwise disjoint.*

7.2. The commutant \mathcal{A}_{pq} . Recall that the representation T_{pq} is an inductive limit of finite dimensional representations of the groups $G(n)$ in the spaces $H^n \cap H(T_{pq}) \subset \text{Reg}^n$. Since

$$(v_{pq}, \xi_\lambda) = f_{pq}(\lambda) \neq 0 \quad \text{for } \lambda \in \mathbb{Y}(p, q),$$

the projection $v_{pq}^{(n)}$ of the vector v_{pq} onto subspace $H^n \cap H(T_{pq})$ is a cyclic vector, for every n . If one of the numbers p, q vanishes, we have $v_{pq}^{(n)} \equiv v_{pq}$ and hence v_{pq} is obviously a cyclic vector. But if $v_{pq}^{(n)} \neq v_{pq}$, the fact that each vector $v_{pq}^{(n)}$ is cyclic in the corresponding representation of the subgroup $G(n)$ does not formally imply that the limiting vector is cyclic, too. See subsection 7.8. for a counterexample.

In order to prove that the vector v_{pq} is cyclic we study the commutant of T_{pq} , making use of Proposition 5.3.1 and Theorem 5.5.1. We shall show that the matrix element $((\cdot)v_{pq}, v_{pq})$ provides a *faithful* state on the commutant, which implies the cyclicity immediately. Unfortunately, the proof turns out to be rather long.

Fix a block $T_{pq} \subset T_k$, where $k = p - q$, and assume that $p \geq 1, q \geq 1$ (if one of the numbers p, q vanishes, the claim of the Theorem 7.1.1 is trivial).

Set $\mathbb{Y}_n(p, q) = \mathbb{Y}_n \cap \mathbb{Y}(p, q)$; this is the n th level of the graph $\mathbb{Y}(p, q)$. It is not empty, starting with $n = pq > 0$. The graph $\mathbb{Y}(p, q)$ can be identified with the direct product of the graphs $\mathbb{Y}(p, 0)$ and $\mathbb{Y}(0, q)$.

Denote by \mathcal{A}_{pq} the commutant of the representation T_{pq} . According to Proposition 5.3.1, there is a linear isomorphism $\mathcal{A}_{pq} \rightarrow \tilde{\mathcal{A}}_{pq}$, where $\tilde{\mathcal{A}}_{pq}$ is the Banach space of complex-valued bounded functions $A(\lambda)$ on $\mathbb{Y}(p, q)$, satisfying the condition

$$A(\lambda) = \sum_{\nu \searrow \lambda} \tilde{p}_z(\lambda, \nu) A(\nu), \quad \lambda \in \mathbb{Y}(p, q),$$

with the norm

$$\|A\| = \sup_{\lambda} |A(\lambda)|.$$

According to Theorem 5.5.1, the function $\tilde{p}_z(\lambda, \nu)$ has the form

$$\tilde{p}_z(\lambda, \nu) = p_z(\lambda, \nu) = \frac{|z + c_{\lambda, \nu}|^2}{|z|^2 + n} \cdot \frac{\dim \nu}{(n+1) \dim \lambda}.$$

It will be important for us that $p_z(\lambda, \nu) \neq 0$ for all couples $\lambda \nearrow \nu$ in the graph $\mathbb{Y}(p, q)$.

Denote by \mathcal{A}_{pq}^+ the cone of nonnegative Hermitian operators in \mathcal{A}_{pq} . Its image with respect to the isomorphism $\mathcal{A}_{pq} \rightarrow \tilde{\mathcal{A}}_{pq}$ is contained in the cone $\tilde{\mathcal{A}}_{pq}^+ \subset \tilde{\mathcal{A}}_{pq}$ generated by the functions $A \in \tilde{\mathcal{A}}_{pq}$ with nonnegative values. One can show that the image actually coincides with $\tilde{\mathcal{A}}_{pq}^+$, but we do not need this fact.

7.3. A faithful state on the algebra \mathcal{A}_{pq} . Denote by λ_{\min} the rectangular diagram of size $p \times q$; this is the only vertex of the graph $\mathbb{Y}(p, q)$ at the level pq .

Lemma 7.3.1. *The linear functional*

$$\eta : A \mapsto A(\lambda_{\min}), \quad A \in \tilde{\mathcal{A}}_{pq}$$

provides a faithful trace on the algebra \mathcal{A}_{pq} .

Proof. Clearly, the functional η equals 1 at the function $A(\lambda) \equiv 1$ (which determines the identity of \mathcal{A}_{pq}), has norm 1 and is nonnegative on the cone \mathcal{A}_{pq}^+ . Hence, it defines a state on the algebra \mathcal{A}_{pq} . It remains to prove that this state is faithful. To do this we shall show that $\eta(A) > 0$ for every non-zero function $A(\lambda)$ in $\tilde{\mathcal{A}}_{pq}^+$.

Define a “weight function” $\mathfrak{M}_{pq}(\lambda)$ on the graph $\mathbb{Y}(p, q)$ by the recurrence relation

$$\mathfrak{M}_{pq}(\nu) = \sum_{\lambda: \lambda \nearrow \nu} \mathfrak{M}_{pq}(\lambda) p_z(\lambda, \nu), \quad |\nu| > pq$$

and the initial condition

$$\mathfrak{M}_{pq}(\lambda_{\min}) = 1.$$

Denote by $\mathfrak{M}_{pq}^{(n)}$ the measure on $\mathbb{Y}_n(p, q)$, $n \geq pq$, with the weights $\mathfrak{M}_{pq}(\lambda)$ at the vertices $\lambda \in \mathbb{Y}_n(p, q)$. Since $p_z(\lambda, \nu)$ has the property

$$\sum_{\nu \searrow \lambda} p_z(\lambda, \nu) = 1,$$

all $\mathfrak{M}_{pq}^{(n)}$ are probability measures.

By virtue of the main relation for the functions $A(\cdot) \in \tilde{\mathcal{A}}_{pq}$, the expression

$$\eta_n(A) = \langle \mathfrak{M}_{pq}^{(n)}, A \rangle = \sum_{\lambda \in \mathbb{Y}_n(p, q)} \mathfrak{M}_{pq}(\lambda) A(\lambda), \quad n \geq pq,$$

does not depend on n . Hence, it coincides with $\eta_{pq}(A) = \eta(A)$.

Since $p_z(\lambda, \nu) > 0$, all the weights $\mathfrak{M}_{pq}(\lambda)$ are strictly positive. Let now $A(\cdot)$ be a non-zero function from $\tilde{\mathcal{A}}_{pq}^+$. Then there exists $\lambda \in \mathbb{Y}(p, q)$, such that $A(\lambda) > 0$. If $n = |\lambda|$, then $\langle \mathfrak{M}_{pq}^{(n)}, A \rangle > 0$, hence we conclude that $\eta(A) = \eta_n(A) > 0$. \square

7.4. Evaluation of \mathfrak{M}_{pq} . As in §6 above, we associate with a diagram $\lambda \in \mathbb{Y}(p, q)$ a couple of diagrams (λ^+, λ^-) , and in order to simplify the notation we set $a_i = \lambda_i^+$, $b_r = \lambda_r^-$ for $1 \leq i \leq p$, $1 \leq r \leq q$.

Lemma 7.4.1. The “weight function” $\mathfrak{M}_{pq}(\lambda)$ introduced in the proof of Lemma 7.2.1 can be given by the explicit formula

$$\mathfrak{M}_{pq}(\lambda) = \frac{C(p, q)}{(n - pq + 1)_{p^2 + q^2 - pq - 1}} \times \frac{\prod_{1 \leq i < j \leq p} (a_i - a_j + j - i)^2 \prod_{1 \leq r < s \leq q} (b_r - b_s + s - r)^2}{\prod_{i, r} (a_i + b_r + p - i + q - r + 1)},$$

where $n = |\lambda|$ and

$$C(p, q) = \frac{(p^2 + q^2 - pq - 1)! \prod_{i, r} (p - i + q - r + 1)}{(\prod_i (p - i)! \prod_r (q - r)!)^2}.$$

Proof. Recall that $\mathfrak{M}_{pq}(\lambda)$ satisfies the recurrence relation

$$\mathfrak{M}_{pq}(\nu) = \sum_{\lambda \nearrow \nu} \frac{(p - q + c_{\lambda\nu})^2}{(p - q)^2 + n} \frac{\dim \nu}{(n + 1) \dim \lambda} \mathfrak{M}_{pq}(\lambda),$$

where $|\nu| = n + 1 > pq$.

Set in this formula

$$\mathfrak{M}_{pq}(\lambda) = \frac{\prod_{b \in \lambda \setminus \lambda_{\min}} (p - q + c(b))^2}{((p - q)^2 + n - 1)!} \frac{\dim \lambda}{n!} \mathfrak{M}'_{pq}(\lambda)$$

with a new unknown function $\mathfrak{M}'_{pq}(\lambda)$, $n = |\lambda|$. Then for $\mathfrak{M}'_{pq}(\lambda)$ we get the recurrence relation

$$\mathfrak{M}'_{pq}(\nu) = \sum_{\lambda \nearrow \nu} \mathfrak{M}'_{pq}(\lambda).$$

Its solution has the form

$$\mathfrak{M}'_{pq}(\lambda) = \text{const} \cdot \text{Dim } \lambda,$$

where $\text{Dim } \lambda$ stands for the number of paths, in the graph $\mathbb{Y}(p, q)$, going from λ_{\min} to λ (the dimension function of the graph $\mathbb{Y}(p, q)$).

Since the graph $\mathbb{Y}(p, q)$ is isomorphic to the product of the graphs $\mathbb{Y}(p, 0) \times \mathbb{Y}(q, 0)$ (with the empty diagrams \emptyset added to $\mathbb{Y}(p, 0)$ and to $\mathbb{Y}(0, q)$), we get

$$\text{Dim } \lambda = \frac{(n - pq)!}{|\lambda^+|! |\lambda^-|!} \dim \lambda^+ \dim \lambda^-.$$

It follows that

$$\mathfrak{M}_{pq}(\lambda) = \text{const} \frac{\prod_{b \in \lambda \setminus \lambda_{\min}} (p - q + c(b))^2}{((p - q)^2 + n - 1)!} \frac{\dim \lambda}{|\lambda|!} \frac{\dim \lambda^+}{|\lambda^+|!} \frac{\dim \lambda^-}{|\lambda^-|!} (n - pq)!.$$

Substitute here the explicit expressions:

$$\prod_{b \in \lambda \setminus \lambda_{\min}} (p - q + c(b))^2 = \left(\prod_{i=1}^p \frac{(a_i + p - i)!}{(p - i)!} \prod_{r=1}^q \frac{(b_r + q - r)!}{(q - r)!} \right)^2$$

(this is readily checked),

$$\frac{\dim \lambda}{|\lambda|!} = \frac{\prod_{1 \leq i < j \leq p} (a_i - a_j + j - i) \prod_{1 \leq r < s \leq q} (b_r - b_s + s - r)}{\prod_{1 \leq i \leq p} (a_i + p - i)! \prod_{1 \leq r \leq q} (b_r + q - r)! \prod_{i,r} (a_i + b_r + p - i + q - r + 1)},$$

(this can be derived from the hook formula in the same way as formula (2.7) in Olshanski [Ol5]), and

$$\frac{\dim \lambda^+}{|\lambda^+|!} = \frac{\prod_{1 \leq i < j \leq p} (a_i - a_j + j - i)}{\prod_{1 \leq i \leq p} (a_i + p - i)!}, \quad \frac{\dim \lambda^-}{|\lambda^-|!} = \frac{\prod_{1 \leq r < s \leq q} (b_r - b_s + s - r)}{\prod_{1 \leq r \leq q} (b_r + q - r)!}$$

(this is the well-known Frobenius formula, see, e.g., Macdonald [Mac, Ex. I.7.6]). Then we get the desired expression, where the constant const can be found from the condition $\mathfrak{M}_{pq}(\lambda_{\min}) = 1$. \square

7.5. The limit of the measures $\mathfrak{M}_{pq}^{(n)}$. We shall need two technical lemmas.

Lemma 7.5.1. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{\lambda \in \mathbb{Y}_n(p,q); a_p + b_q \leq \delta n} \mathfrak{M}_{pq}(\lambda) \leq \varepsilon,$$

for all n large enough.

Proof will be given in §7.7.

Lemma 7.5.2. *The function*

$$\frac{\prod_{1 \leq i < j \leq p} (\alpha_i - \alpha_j)^2 \prod_{1 \leq r < s \leq q} (\beta_r - \beta_s)^2}{\prod_{i,r} (\alpha_i + \beta_r)}$$

on the simplex $\Omega(p, q)$ is integrable with respect to Lebesgue measure $d\omega$.

Proof will be given in §7.8.

Consider now the embedding $\bar{\iota}_n : \mathbb{Y}_n(p, q) \rightarrow \Omega(p, q)$ introduced in §6.3. Let $\widehat{\mathfrak{M}}_{pq}^{(n)} = \bar{\iota}_n(\mathfrak{M}_{pq}^{(n)})$ be the push-forward of the probability measure $\mathfrak{M}_{pq}^{(n)}$; this is a probability measure on $\Omega(p, q)$.

Lemma 7.5.3. *As $n \rightarrow \infty$, the measures $\widehat{\mathfrak{M}}_{pq}^{(n)}$ on $\Omega(p, q)$ weakly converge to a certain probability measure $\widehat{\mathfrak{M}}_{pq}^{(\infty)}$. The measure $\widehat{\mathfrak{M}}_{pq}^{(\infty)}$ is absolutely continuous with respect to Lebesgue measure $d\omega$ on $\Omega(p, q)$, and has the density*

$$C(p, q) \frac{\prod_{1 \leq i < j \leq p} (\alpha_i - \alpha_j)^2 \prod_{1 \leq r < s \leq q} (\beta_r - \beta_s)^2}{\prod_{i,r} (\alpha_i + \beta_r)},$$

where the constant is the same as in Lemma 7.4.1.

Proof. Let $\lambda \in \mathbb{Y}_n(p, q)$ and $\alpha; \beta = \bar{t}_n(\lambda)$. It follows from the expression for $\mathfrak{M}_{pq}(\lambda)$ in Lemma 7.4.1 that

$$\mathfrak{M}_{pq}^{(n)}(\lambda) = C(p, q) n^{-(p+q-1)} (1 + O(1/n)) \times \frac{\prod_{1 \leq i < j \leq p} (\alpha_i - \alpha_j + O(1/n))^2 \prod_{1 \leq r < s \leq q} (\beta_r - \beta_s + O(1/n))^2}{\prod_{i, r} (\alpha_i + \beta_r + O(1/n))}.$$

Given $\delta > 0$, we split the simplex $\Omega(p, q)$ into the union of two subsets,

$$\Omega(p, q) = \Omega_{\geq \delta}(p, q) \cup \Omega_{< \delta}(p, q),$$

determined by the conditions $\alpha_p + \beta_q \geq \delta$ and $\alpha_p + \beta_q < \delta$, respectively. On the set $\Omega_{\geq \delta}(p, q)$ the denominator of the expression for $\mathfrak{M}_{pq}^{(n)}(\lambda)$ is bounded from below, uniformly in n . This implies the weak convergence of measures

$$\widehat{\mathfrak{M}}_{pq}^{(n)} \Big|_{\Omega_{\geq \delta}(p, q)} \longrightarrow \widehat{\mathfrak{M}}_{pq}^{(\infty)} \Big|_{\Omega_{\geq \delta}(p, q)}.$$

By Lemma 7.5.1, the total mass of the set $\Omega_{< \delta}(p, q)$ with respect to $\widehat{\mathfrak{M}}_{pq}^{(n)}$ can be made arbitrarily small (uniformly in n), provided that δ is chosen sufficiently small, and n is large enough. By Lemma 7.5.2, the mass of the set $\Omega_{< \delta}(p, q)$ with respect to $\widehat{\mathfrak{M}}_{pq}^{(\infty)}$ also tends to 0, together with δ . This implies that $\widehat{\mathfrak{M}}_{pq}^{(n)}$ weakly converges to $\widehat{\mathfrak{M}}_{pq}^{(\infty)}$ on the entire simplex $\Omega(p, q)$. \square

Lemma 7.5.4. *The state*

$$A \mapsto \frac{(Av_{pq}, v_{pq})}{(v_{pq}, v_{pq})}$$

on the algebra \mathcal{A}_{pq} is faithful.

Proof. We have to prove that if $A \in \mathcal{A}_{pq}^+$, $A \neq 0$, then $(Av_{pq}, v_{pq}) > 0$. Set

$$A^{(n)} = P_n A P_n, \quad v_{pq}^{(n)} = P_n v_{pq}.$$

If $n \rightarrow \infty$, the operator $P_n A P_n$ converges to A strongly, hence

$$(Av_{pq}, v_{pq}) = \lim_{n \rightarrow \infty} (P_n A P_n v_{pq}, v_{pq}) = \lim_{n \rightarrow \infty} (A^{(n)} v_{pq}^{(n)}, v_{pq}^{(n)}).$$

According to Proposition 5.3.1,

$$A^{(n)} = \sum_{\lambda \in \mathbb{Y}_n(p, q)} A(\lambda) P(\lambda),$$

where $A(\lambda)$ is the function in $\tilde{\mathcal{A}}_{pq}^+$ corresponding to the operator A . On the other hand,

$$v_{pq}^{(n)} = \sum_{\lambda \in \mathbb{Y}_n(p, q)} f_{pq}(\lambda) \xi_\lambda$$

by definition of the vector v_{pq} . Therefore,

$$(A^{(n)}v_{pq}^{(n)}, v_{pq}^{(n)}) = \sum_{\lambda \in \mathbb{Y}_n(p,q)} |f_{pq}(\lambda)|^2 A(\lambda) = \sum_{\lambda \in \mathbb{Y}_n(p,q)} A(\lambda) \mathfrak{M}_{pq}(\lambda) \frac{|f_{pq}(\lambda)|^2}{\mathfrak{M}_{pq}(\lambda)}.$$

We have explicit expressions for both $f_{pq}(\lambda)$ and $\mathfrak{M}_{pq}(\lambda)$, see Theorem 6.2.1 and Lemma 7.4.1. These expressions imply that

$$\frac{|f_{pq}(\lambda)|^2}{\mathfrak{M}_{pq}(\lambda)} = c_{pqn} \prod_{i,r} (a_i + b_r + p - i + q - r + 1),$$

where

$$c_{pqn} = \frac{(n - pq + 1)_{p^2+q^2-pq-1}}{x_n^2 \cdot C(p, q)}$$

(we have employed here an equivalent expression for $f_{pq}(\lambda)$, see Lemma 6.3.3).

Since

$$x_n \sim n^{(p^2+q^2-1)/2},$$

we have

$$c_{pqn} \sim \frac{1}{C(p, q) n^{pq}},$$

whence

$$\begin{aligned} \frac{|f_{pq}(\lambda)|^2}{\mathfrak{M}_{pq}(\lambda)} &= (\text{const} + o(1)) \prod_{i,r} \frac{a_i + b_r + p - i + q - r + 1}{n} \\ &= \text{const} \prod_{i,r} (\alpha_i + \beta_r) + o(1), \end{aligned}$$

where we assume that $(\alpha; \beta) = \bar{\iota}_n(\lambda)$.

Let $\mathfrak{N}_{pq}^{(n)}$ denote the measure on $\mathbb{Y}_n(p, q)$ defined by

$$\mathfrak{N}_{pq}^{(n)}(\lambda) = A(\lambda) \mathfrak{M}_{pq}(\lambda), \quad \lambda \in \mathbb{Y}_n(p, q).$$

In the notation introduced in the proof of Lemma 7.3.1,

$$\sum_{\lambda \in \mathbb{Y}_n(p,q)} A(\lambda) \mathfrak{M}_{pq}(\lambda) = \eta_n(A).$$

We have noted there that $\eta_n(A)$ does not depend on n , and is strictly positive. Therefore, the mass of the set $\mathbb{Y}_n(p, q)$ with respect to the measure $\mathfrak{N}_{pq}^{(n)}$ is strictly positive and does not depend on n . Consequently, passing to an appropriate subsequence of indices $n_1 < n_2 < \dots$, we may conclude that the measures $\widehat{\mathfrak{N}}_{pq}^{(n)} := \bar{\iota}_n(\mathfrak{N}_{pq}^{(n)})$ converge weakly on $\Omega(p, q)$ to a certain nonzero measure $\widehat{\mathfrak{N}}_{pq}^{(\infty)}$.

On the other hand, for the weights of the measures $\mathfrak{N}_{pq}^{(n)}$ and $\mathfrak{M}_{pq}^{(n)}$ there is an estimate

$$\mathfrak{N}_{pq}^{(n)}(\lambda) = A(\lambda) \mathfrak{M}_{pq}^{(n)}(\lambda) \leq \|A\| \mathfrak{M}_{pq}^{(n)}(\lambda),$$

hence

$$\widehat{\mathfrak{N}}_{pq}^{(n)} \leq \|A\| \widehat{\mathfrak{M}}_{pq}^{(n)}.$$

By Lemma 7.5.3, the measures $\widehat{\mathfrak{M}}_{pq}^{(n)}$ converge weakly to a measure $\widehat{\mathfrak{M}}_{pq}^{(\infty)}$ which is absolutely continuous with respect to the Lebesgue measure $d\omega$ on the simplex $\Omega(p, q)$. It follows that

$$\widehat{\mathfrak{N}}_{pq}^{(\infty)} \leq \|A\| \widehat{\mathfrak{M}}_{pq}^{(\infty)},$$

which implies that $\widehat{\mathfrak{N}}_{pq}^{(\infty)}$ is absolutely continuous, too.

Return now to the quantity (Av_{pq}, v_{pq}) . We have shown that it can be represented as a limit

$$(Av_{pq}, v_{pq}) = \lim_{n \rightarrow \infty} (A^{(n)} v_{pq}^{(n)}, v_{pq}^{(n)}) = \lim_{n \rightarrow \infty} \text{const} \left\langle \widehat{\mathfrak{N}}_{pq}^{(n)}, \left(\prod_{i,r} (\alpha_i + \beta_r) + o(1) \right) \right\rangle,$$

where $\text{const} > 0$. Hence,

$$(Av_{pq}, v_{pq}) = \text{const} \left\langle \widehat{\mathfrak{N}}_{pq}^{(\infty)}, \prod_{i,r} (\alpha_i + \beta_r) \right\rangle.$$

Since the measure $\widehat{\mathfrak{N}}_{pq}^{(\infty)}$ is absolutely continuous and nonzero, and the function $\prod (\alpha_i + \beta_r)$ on the simplex $\Omega(p, q)$ is continuous and positive almost everywhere, the result is a strictly positive number. \square

7.6. The Cauchy determinant. We shall need a generalization of the classical Cauchy identity

$$\det \left[\frac{1}{x_i + y_j} \right]_{i,j=1}^m = \frac{V(x) V(y)}{\prod_{i,j} (x_i + y_j)},$$

where

$$V(x) = \prod_{1 \leq i < j \leq m} (x_i - x_j),$$

to the case when the numbers of x_i 's and y_j 's are not necessarily equal.

Set

$$x = (x_1, \dots, x_p), \quad y = (y_1, \dots, y_q),$$

and assume (to be definite) that $p \leq q$. We shall denote by the symbol $y = y' \sqcup y''$ an arbitrary decomposition of the set of variables $y = (y_1, \dots, y_q)$ into disjoint subsets of cardinalities $q - p$ and p respectively:

$$y' = (y'_1, \dots, y'_{q-p}) = (y_{i_1}, \dots, y_{i_{q-p}}), \quad y'' = (y''_1, \dots, y''_p) = (y_{j_1}, \dots, y_{j_p}),$$

where

$$i_1 < \dots < i_{q-p}, \quad j_1 < \dots < j_p, \quad \{i_1, \dots, i_{q-p}\} \cup \{j_1, \dots, j_p\} = \{1, \dots, q\}.$$

Set $\text{sgn}(y', y'') = \pm 1$, where the sign plus/minus is taken depending on parity/impairity of the number of inversions in the permutation $(i_1, \dots, i_{q-p}, j_1, \dots, j_p)$ of the numbers $1, \dots, q$.

Lemma 7.6.5. *The following formula generalizes the Cauchy determinant:*

$$\frac{V(x_1, \dots, x_p) V(y_1, \dots, y_q)}{\prod_{i,j} (x_i + y_j)} = \sum_{y' \sqcup y'' = y} \text{sgn}(y', y'') V(y'_1, \dots, y'_{q-p}) \det \left[\frac{1}{x_i + y''_j} \right]_{i,j=1}^p.$$

Proof. One can easily derive this identity from the classical Cauchy identity by replacing the variables x_1, \dots, x_p with x'_1, \dots, x'_p , adding a new group of variables x'_1, \dots, x'_{q-p} , writing down the Cauchy identity in terms of the variables $(\varepsilon x'_1, \dots, \varepsilon x'_{q-p}; x''_1, \dots, x''_p)$ and y_1, \dots, y_q , and then applying the Laplace rule while ε goes to 0. \square

Another proof. The argument given below is similar to the well-known proof of the classical Cauchy identity.

Denote the right-hand side of the identity in question by $A(x, y)$. Then the product $A(x, y) \Pi(x_i + y_j)$ is a polynomial. It suffices to check the following three claims:

- (i) $A(x, y)$ is skew symmetric, separately in x and in y .
- (ii) $\deg A(x, y) = \deg V(x) + \deg V(y) - \deg \Pi(x_i + y_j)$.
- (ii) Let us order the variables as $x_1 > \dots > x_p > y_1 > \dots > y_q$ and consider the corresponding order on the monomials; then the leading term in the expansion of $A(x, y) \Pi(x_i + y_j)$ is the same as that for $V(x) V(y)$.

Now we have:

- (i) The skew symmetry with respect to x is evident. Let us show that $A(x, y)$ changes the sign upon the elementary transposition $y_j \leftrightarrow y_{j+1}$, where $j = 1, \dots, q-1$. If the variables y_j, y_{j+1} enter the same group, y' or y'' , then the corresponding term is already skew symmetric. Consider now the terms for which y_j and y_{j+1} belong to different groups. Those terms split into pairs: in each pair the terms are switched by the transposition $y_j \leftrightarrow y_{j+1}$ and the corresponding signs are opposite. Thus, the skew symmetry follows.

- (ii) This is trivial.

- (iii) It is readily verified that the leading monomial comes from the partition $y' = (y_1, \dots, y_{q-p}), y'' = (y_{q-p+1}, \dots, y_q)$. \square

7.7. Proof of Lemma 7.5.1. Without loss of generality we may assume that $p \leq q$. Set

$$x_i = a_i + p - i, \quad 1 \leq i \leq p; \quad y_j = b_j + q - j, \quad 1 \leq j \leq q.$$

Taking into account the explicit formula for $\mathfrak{M}_{pq}(\lambda)$ (Lemma 7.4.1) we have to prove the estimate

$$\frac{1}{n^{p^2-pq+q^2-1}} \sum_{x,y} \frac{V^2(x) V^2(y)}{\prod_{i,j} (x_i + y_j + 1)} = O(\delta),$$

summed over the integer vectors $x \in \mathbb{Z}_+^p, y \in \mathbb{Z}_+^q$ satisfying the conditions

$$\begin{aligned} x_1 > \dots > x_p &\geq 0, & y_1 > \dots > y_q &\geq 0, & x_p + y_q &\leq \delta n \\ x_1 + \dots + x_p + y_1 + \dots + y_q &= n + \text{const}, \end{aligned} \tag{7-6-1}$$

where “const” is an integer not depending on n .

Apply the identity of Lemma 7.6.1. Since all variables are of order $O(n)$, we have

$$V(x)V(y)V(y') = O\left(n^{\frac{1}{2}(p(p-1)+q(q-1)+(q-p)(q-p-1))}\right) = O\left(n^{p^2-pq+q^2-q}\right).$$

Using this and expanding the determinant in the right hand side of the identity in question, we reduce the problem to the following bound:

$$\frac{1}{n^{q-1}} \sum_{x,y} \frac{1}{(x_1 + y''_{\sigma(1)} + 1) \dots (x_p + y''_{\sigma(p)} + 1)} = O(\delta),$$

where a splitting of variables $y' \sqcup y'' = y$ is fixed, σ is a fixed permutation of the indices $1, \dots, p$, and summation is again taken under the conditions (7-6-1).

There are three possibilities:

- (i) y'' contains y_q , and $y''_{\sigma(p)} = y_q$;
- (ii) y'' contains y_q , but $y''_{\sigma(p)} \neq y_q$;
- (iii) y'' does not contain y_q .

The case (iii) reduces to that of (i), since replacing $y''_{\sigma(p)}$ with $y_q < y''_{\sigma(p)}$ only increases the sum.

The case (ii) can also be reduced to (i). Indeed, there exists $i < p$ such that $y''_{\sigma(i)} = y_q$. We can now change σ by switching $y''_{\sigma(i)}$ and $y''_{\sigma(p)}$. This can be done by virtue of the inequality

$$\frac{1}{(A_1 + B_2)(A_2 + B_1)} < \frac{1}{(A_1 + B_1)(A_2 + B_2)},$$

(correct for $A_1 > A_2 > 0$, $B_1 > B_2 > 0$) which we apply to

$$A_1 = x_i + \frac{1}{2}, \quad A_2 = x_p + \frac{1}{2}, \quad B_1 = y''_{\sigma(p)} + \frac{1}{2}, \quad B_2 = y''_{\sigma(i)} + \frac{1}{2}.$$

Hence, we are left with the case (i), where $y''_{\sigma(p)} = y_q$. Let us relax the system of restrictions (7-6-1) by removing from it the inequalities $x_1 > \dots > x_p$, $y_1 > \dots > y_q$ (but we still assume that all variables are nonnegative integers). This will result in a bigger amount of arrays x, y , hence the sum will increase, too.

Set

$$w_1 = x_1 + y''_{\sigma(1)}, \quad \dots, \quad w_{p-1} = x_{p-1} + y''_{\sigma(p-1)}, \quad w_p = x_p + y''_{\sigma(p)} = x_p + y_p.$$

Note that for every w_i there are precisely $w_i + 1$ ways to represent it as a sum of two nonnegative terms. Therefore, our bound reduces to the following one: the number of vectors $(y'_1, \dots, y'_{q-p}, w_1, \dots, w_p) \in \mathbb{Z}_+^q$ such that

$$y'_1 + \dots + y'_{q-p} + w_1 + \dots + w_p = n + \text{const}, \quad w_p \leq \delta n$$

is of order $O(\delta)n^{q-1}$. This bound is readily verified.

7.8. Proof of Lemma 7.5.2. We shall prove the analogous claim for the same density on the larger simplex $\tilde{\Omega}(p, q)$, obtained by removing the restrictions $\alpha_1 \geq \dots \geq \alpha_p$, $\beta_1 \geq \dots \geq \beta_q$. In other words, $\tilde{\Omega}(p, q)$ is just the standard $(p + q - 1)$ -dimensional simplex. Without loss of generality we may assume that $p \leq q$. By the Cauchy type identity of Lemma 7.6.5 and taking into account the trivial bounds $\alpha_i \leq 1$, $\beta_j \leq 1$ we are reduced to the following claim:

$$\int \frac{dw}{(\alpha_1 + \beta_1) \dots (\alpha_p + \beta_p)} < \infty,$$

where dw is the Lebesgue measure on the standard $(p + q - 1)$ -dimensional simplex $\sum \alpha_i + \sum \beta_j = 1$, $\alpha_i \geq 0$, $\beta_j \geq 0$, the domain of integration. Project our simplex onto a $(q - 1)$ -dimensional simplex by applying the map

$$(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q) \longmapsto (\alpha_1 + \beta_1, \dots, \alpha_p + \beta_p; \beta_{p+1}, \dots, \beta_q).$$

Under this projection, the push-forward of the measure with density

$$\frac{1}{(\alpha_1 + \beta_1) \dots (\alpha_p + \beta_p)}$$

is simply the Lebesgue measure (this is a particular case of a more general fact on the behavior of Dirichlet measures under projections of simplices), which completes the proof of the lemma.

This completes the proofs of Theorems 7.1.1, 7.1.2.

7.9. Example of a noncyclic vector with cyclic projections. One can think that the proof of the cyclicity of the vectors v_{pq} given above is a little bit too involved. Surely, one would try to simplify it. But the claim is not entirely trivial, and the goal of the present addendum is show that by an example.

We shall provide a representation $T = \varinjlim T_n$ of a group $G = \varinjlim G_n$, and a vector $\xi \in H(T)$, such that, for each n , the projection ξ_n of the vector ξ onto the subspace $H(T_n)$ is cyclic for T_n , though the vector ξ itself is not.

Set $G_n = \mathbb{Z}_2^n$, so that $G = \bigoplus_1^\infty \mathbb{Z}_2$. The dual group to G_n is again \mathbb{Z}_2^n , while the dual group to G is a compact group, $\hat{G} = \prod_1^\infty \mathbb{Z}_2$. With arbitrary finite Borel measure σ on \hat{G} one can associate a unitary representation T of the group G acting in the Hilbert space $H = L^2(\hat{G}, \sigma)$ by the formula

$$(T(g)f)(\chi) = \chi(g) f(\chi), \quad g \in G, \quad \chi \in \hat{G}, \quad f \in H.$$

Denote by $H_n \subset H$ the subspace of functions $f(\chi)$ depending on $\chi_n := \chi|_{G_n}$ only, and by T_n the natural representation of the group G_n in H_n . The representation T coincides with $\varinjlim T_n$. The space H_n can be identified with $L^2(\mathbb{Z}_2^n, \sigma^{(n)})$, where $\sigma^{(n)}$ is the image of the measure σ under the projection $\hat{G} \rightarrow \mathbb{Z}_2^n$ (taking of the first n coordinates). If $\xi = f(\cdot) \in H$, then the vector $\xi_n \in H_n = L^2(\mathbb{Z}_2^n, \sigma^{(n)})$ can be obtained by integrating f along the fibers of the projection $G \rightarrow \mathbb{Z}_2^n$.

For $p \in (0, 1)$ denote by σ_p the Bernoulli measure on $\hat{G} = \prod_1^\infty \mathbb{Z}_2 = \prod_1^\infty \{0, 1\}$ with the weights p and $1 - p$ at the points 0 and 1 respectively. By virtue of the law of large numbers, σ_p is supported by the set $X_p \subset \prod_1^\infty \{0, 1\}$ formed by 0-1 sequences with the limiting frequency of 0's equal to p .

Take two distinct numbers $p, p' \in (0, 1)$ and set $\sigma = \sigma_p + \sigma_{p'}$. The measure σ is supported by the union of two disjoint sets $X_p, X_{p'}$ each of which has measure 1. Take for ξ the characteristic function of the set X_p . Clearly, ξ is not cyclic, since its cyclic span is a proper subspace generated by the functions in $L^2(\widehat{G}, \sigma)$ supported by X_p .

On the other hand, the vector ξ_n , as a function on \mathbb{Z}_2^n , coincides with the Radon–Nikodym derivative $\sigma_p^{(n)} / (\sigma_p^{(n)} + \sigma_{p'}^{(n)})$, hence is a strictly positive function. Therefore, ξ_n is a cyclic vector for T_n for any n .

8. DISJOINTNESS

8.1. The results. Our aim is to prove the following result

Theorem 8.1.1. *If z ranges over the upper half-plane $\Im z \geq 0$, then the representations T_z are pairwise disjoint.*

The assumption $\Im z \geq 0$ is introduced because $T_z \sim T_{\bar{z}}$. The definition of disjoint representations is given in §9...

Let σ_z the spectral measure of the character χ_z (§4.1). We shall deduce Theorem 8.1.1 from the following result.

Theorem 8.1.2. *Assume z ranges over the set $\{z \in \mathbb{C} \setminus \mathbb{Z}, \Im z \geq 0\}$.*

- (i) *The measures σ_z are pairwise disjoint.*
- (ii) *Each of the faces $\Omega(p, q) \subset \Omega$ is a null set with respect to σ_z .*

Derivation of Theorem 8.1.1 from Theorem 8.1.2. Let z_1, z_2 be two distinct complex numbers from the upper half-plane $\Im z \geq 0$. We have to prove that T_{z_1} and T_{z_2} are disjoint. Assume first that both z_1 and z_2 are not integers. We know that if $z \in \mathbb{C} \setminus \mathbb{Z}$ then the distinguished spherical vector ξ_0 is a cyclic vector; hence the measure σ_z determines the representation T_z entirely. According to claim (i) of Theorem 8.1.2, the measures σ_{z-1} and σ_{z_2} are disjoint; therefore the representations T_{z_1} and T_{z_2} are disjoint, too. Assume now that $z_1 \in \mathbb{C} \setminus \mathbb{Z}$ while $z_2 \in \mathbb{Z}$. According to Theorem 7.1.2, the representation T_{z_2} decomposes into a direct sum of representations labelled by the faces $\Omega(p, q)$, $p - q = z_2$. By virtue of claim (ii) of Theorem 8.1.2, T_{z_1} and T_{z_2} are disjoint. Finally, when z_1, z_2 are two distinct integers, the disjointness of the representations was pointed out in Corollary 7.1.3. \square

Now we proceed to the proof of Theorem 8.1.2.

8.2. Reduction to central measures. There is a one-to-one correspondence $\sigma \leftrightarrow \widetilde{M}$ between probability measures σ on the Thoma simplex Ω and central probability measures \widetilde{M} on the path space \mathcal{T} , see §9...

Lemma 8.2.1. *Let σ_1 and σ_2 be two probability measures on Ω , and let \widetilde{M}_1 and \widetilde{M}_2 be the corresponding central measures on \mathcal{T} . Then σ_1, σ_2 are disjoint if and only if \widetilde{M}_1 and \widetilde{M}_2 are disjoint.*

Proof. First, introduce a notation. Given two finite (not necessarily normalized) Borel measures ν_1, ν_2 on a Borel space, let us denote by $\nu_1 \wedge \nu_2$ their greatest lower bound. Its existence can be verified as follows. Let f_1 and f_2 be the Radon–Nikodym derivatives of ν_1 and ν_2 with respect to $\nu_1 + \nu_2$, then we set $\nu_1 \wedge \nu_2 = \min(f_1, f_2)(\nu_1 + \nu_2)$. Observe that ν_1 and ν_2 are disjoint if and only if $\nu_1 \wedge \nu_2 = 0$.

Next, observe that the correspondence $\sigma \leftrightarrow \widetilde{M}$ can be extended to not necessarily normalized measures.

Now we can proceed to the proof. In one direction the implication is trivial. Namely, if σ_1 and σ_2 are not disjoint, then $\sigma_1 \wedge \sigma_2$ is a nonzero measure. Let \widetilde{M} be the corresponding central measure. From the integral representation of Theorem 9.7.2 it follows that $\widetilde{M} \leq \widetilde{M}_1$ and $\widetilde{M} \leq \widetilde{M}_2$, so that $\widetilde{M}_1 \wedge \widetilde{M}_2 \neq 0$, whence \widetilde{M}_1 and \widetilde{M}_2 are not disjoint.

In the opposite direction, assume that \widetilde{M}_1 and \widetilde{M}_2 are not disjoint, so that $\widetilde{M}_1 \wedge \widetilde{M}_2$ is nonzero. We claim that $\widetilde{M}_1 \wedge \widetilde{M}_2$ is a central measure. Indeed, this follows from the characterization of central measures as invariant measures with respect to a countable group action, as explained in Proposition 9.4.1. Now, let σ be the measure on Ω corresponding to $\widetilde{M}_1 \wedge \widetilde{M}_2$. It is a nonzero measure. Next, since $\widetilde{M}_1 \wedge \widetilde{M}_2 \leq \widetilde{M}_1$ and $\widetilde{M}_1 \wedge \widetilde{M}_2 \leq \widetilde{M}_2$, we also have $\sigma \leq \sigma_1$, $\sigma \leq \sigma_2$. (Indeed, this claim can be restated as follows: if $\widetilde{M}, \widetilde{M}'$ are two central probability measures such that $\widetilde{M} \leq \text{const } \widetilde{M}'$ then the same inequality holds for the corresponding spectral measures on Ω , and the latter claim can be checked by applying Theorem 9.7.3 or Theorem 9.7.4.) Therefore, σ_1 and σ_2 are not disjoint. \square

8.3. Proof of claim (ii) of Theorem 8.1.2. Let $p, q \in \mathbb{Z}_+$ be not equal to 0 simultaneously. Denote by $\Gamma(p, q)$ the set

$$\Gamma(p, q) = \{(i, j) \mid 1 \leq i \leq p, \quad j = 1, 2, \dots\} \cup \{(i, j) \mid 1 \leq j \leq q, \quad i = 1, 2, \dots\},$$

(a “fat hook”) and by $\mathcal{T}(p, q)$ the set of paths $\tau = (\tau_n) \in \mathcal{T}$ such that $\tau_n \subset \Gamma(p, q)$ for all n .

Lemma 8.3.1. *If a measure σ is supported by $\Omega(p, q) \subset \Omega$, then the corresponding central measure \widetilde{M} is supported by $\mathcal{T}(p, q) \subset \mathcal{T}$.*

Proof. Let $(M^{(n)})$ be the coherent system corresponding to σ . We have (see §9...)

$$M^{(n)}(\lambda) = \dim \lambda \int_{\omega \in \Omega(p, q)} \widetilde{s}_\lambda(\omega) \sigma(d\omega),$$

where $\widetilde{s}_\lambda(\omega)$ is the supersymmetric Schur polynomial $s_\lambda(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q)$. It is known (Macdonald, [Mac, Example I.5.23 (a)]) that this polynomial vanishes unless λ belongs to $\Gamma(p, q)$. Thus, for each n , $M^{(n)}$ vanishes outside $\Gamma(p, q) \cap \mathbb{Y}_n$. This is equivalent to the statement of the lemma. \square

Lemma 8.3.2. *Let $z \in \mathbb{C} \setminus \mathbb{Z}$ and $p, q \in \mathbb{Z}_+$ be fixed. We assume that $p + q > 0$ thus excluding the case $p = q = 0$. Let $p_z(\lambda, \nu)$ be the coherent system M_z .*

There exists $\varepsilon > 0$ depending on z, p, q only, with the following property. If $\lambda \subset \Gamma(p, q)$ is an arbitrary Young diagram such that the set $\nu = \lambda \cup \{(p+1, q+1)\}$ is also a diagram, then

$$p_z(\lambda, \nu) \geq \varepsilon/n, \quad n = |\lambda|.$$

Proof. Since the content of the box ν/λ is $q - p$, we have

$$p_z(\lambda, \nu) = \frac{|z + q - p|^2}{|z|^2 + n} \cdot \frac{\dim \nu}{(n+1) \dim \lambda}.$$

The first factor can be easily estimated from below: since $z \notin \mathbb{Z}$, there exists $\varepsilon_1 > 0$ depending on z only, such that

$$\frac{|z + q - p|^2}{|z|^2 + n} \geq \frac{\varepsilon_1}{n}.$$

Now consider the second factor. It follows from the hook formula, that

$$\frac{\dim \nu}{(n+1) \dim \lambda} = \prod_b \frac{h(b)}{h(b) + 1},$$

where the product is taken over the boxes $b \in \lambda$ such that either the arm or the leg of b contains the box $(p+1, q+1)$. There are exactly $p+q$ such boxes b , namely

$$(p+1, j), \quad 1 \leq j \leq q; \quad (i, q+1), \quad 1 \leq i \leq p.$$

Therefore, there is a product of $p+q$ factors of the form $k/(k+1)$, where $k = 1, 2, \dots$. Each of the factors is greater or equal to $1/2$, and the entire product is not less than $2^{-(p+q)}$. This provides the required estimate. \square

Lemma 8.3.3. *Let $z \in \mathbb{C} \setminus \mathbb{Z}$ and let \widetilde{M}_z be the central measure on \mathcal{T} corresponding to the measure σ_z . We have $\widetilde{M}_z(\mathcal{T}(p, q)) = 0$ for all p, q .*

Proof. Denote by $\mathcal{T}'(p, q)$ the set of those paths $\tau \in \mathcal{T}(p, q)$ that are not contained in the smaller sets $\mathcal{T}(p-1, q)$ and $\mathcal{T}(p, q-1)$. It suffices to prove that $\mathcal{T}'(p, q)$ has measure 0 with respect to \widetilde{M}_z .

Let μ be an arbitrary diagram in $\Gamma(p, q)$ that contains those two boxes, set $m = |\mu|$, and denote by $\mathcal{T}(p, q; \mu)$ the set of paths $\tau \in \mathcal{T}(p, q)$ with $\tau_m = \mu$. For any path $\tau = (\tau_n) \in \mathcal{T}'(p, q)$ there exists a number n such that the diagram τ_n contains the boxes $(p+1, q)$ and $(p, q+1)$. Consequently the set $\mathcal{T}'(p, q)$ is the countable sum of the sets of the form $\mathcal{T}(p, q; \mu)$. Thus, it remains to prove that each set $\mathcal{T}(p, q; \mu)$ has measure 0.

It will be convenient to look at the measure \widetilde{M}_z as describing a Markov process with the transition function $p_z(\lambda, \nu)$. Set

$$p_n = \text{Prob}\{\tau_{n+1} \mid \mu \subseteq \tau_n \subset \Gamma(p, q)\}.$$

The measure of the set $\mathcal{T}(p, q; \mu)$ coincides with the probability of the event $\tau_m = \mu$, multiplied by the product of the conditional probabilities $\prod_{n \geq m} p_n$.

By Lemma 8.3.2, we have

$$p_n \leq 1 - \frac{\varepsilon}{n}$$

so that $\prod_{n \geq m} p_n = 0$. \square

Proof of claim (ii) of Theorem 8.1.2. Let σ be the restriction of the measure σ_z to $\Omega(p, q)$, and let us show that $\sigma = 0$. Let \widetilde{M} be the central measure corresponding to σ . According to Lemma 8.2.1, \widetilde{M} is supported by $\mathcal{T}(p, q)$. On the other hand, it follows from Lemma 8.3.3 that $\mathcal{T}(p, q)$ is a null set for \widetilde{M}_z . Since $\widetilde{M} \leq \widetilde{M}_z$, we conclude that $\widetilde{M} = 0$, hence $\sigma = 0$. \square

8.4. Proof of claim (i) of Theorem 8.1.2. Recall that if $z \in \mathbb{C} \setminus \mathbb{Z}$, then the measure $M_z^{(n)}$ has nonzero weights $M_z^{(n)}(\lambda)$ for all $\lambda \in \mathbb{Y}_n$, and we have an explicit formula for $M_z^{(n)}(\lambda)$, see Theorem 4.1.1.

Fix two distinct numbers z_1, z_2 in the upper half-plane $\Im z \geq 0$, which are not integers. We have to prove that the spectral measures σ_{z_1} and σ_{z_2} are disjoint. By virtue of Lemma 8.2.1, it suffices to prove that the corresponding central measures \widetilde{M}_{z_1} and \widetilde{M}_{z_2} are disjoint. To simplify the notation, we set $\widetilde{M}_1 = \widetilde{M}_{z_1}$, $\widetilde{M}_2 = \widetilde{M}_{z_2}$. We also denote by $(M_1^{(n)})$ and $(M_2^{(n)})$ the corresponding coherent systems.

Introduce a sequence $\varphi_n(\tau)$ of functions on \mathcal{T} ,

$$\varphi_n(\tau) = \frac{M_2^{(n)}(\tau_n)}{M_1^{(n)}(\tau_n)}, \quad n = 1, 2, \dots, \quad \tau = (\tau_n) \in \mathcal{T}.$$

Let X be the set of paths $\tau \in \mathcal{T}$ such that the sequence $(\varphi_n(\tau))_{n \geq 1}$ converges, as $n \rightarrow \infty$, to a finite nonzero limit. This is a Borel subset of \mathcal{T} .

Lemma 8.4.1. *We have $\widetilde{M}_1(X) = \widetilde{M}_2(X) = 0$.*

Proof. We shall show that X is contained in the union of the sets $\mathcal{T}(p, q)$, so that the claim will follow from Lemma 8.3.3.

Denote by $c_k(\tau)$ the content of the k th box $\tau_k \setminus \tau_{k-1}$. By virtue of the explicit formula for $M_z^{(n)}$ (Theorem 4.1.1), we get

$$\varphi_n(\tau) = \prod_{k=1}^n \left| \frac{z_2 + c_k(\tau)}{z_1 + c_k(\tau)} \right|^2 \frac{|z_1|^2 + k - 1}{|z_2|^2 + k - 1}.$$

Therefore, X consists of those paths τ for which the infinite product

$$\varphi_\infty(\tau) = \prod_{k=1}^{\infty} \left| \frac{z_2 + c_k(\tau)}{z_1 + c_k(\tau)} \right|^2 \frac{|z_1|^2 + k - 1}{|z_2|^2 + k - 1}$$

converges. In particular, the k th factor in the product should go to 1. Since the second fraction in right-hand side converges to 1, as $k \rightarrow \infty$, we conclude that

$$\lim_{k \rightarrow \infty} \left| \frac{z_2 + c_k(\tau)}{z_1 + c_k(\tau)} \right|^2 = 1, \quad \tau \in X.$$

It follows from our assumptions on z_1, z_2 that the equality $|z_2 + c|^2 = |z_1 + c|^2$ may hold for at most one real number c . Indeed, this equation on c describes the set of points c that are equidistant from $-z_1$ and $-z_2$. Since $z_1 \neq z_2$, this set is a line in the complex plane \mathbb{C} , which cannot coincide with the real axis \mathbb{R} , because z_1, z_2 are both in the upper half-plane. Thus, the line is either parallel to \mathbb{R} (then there is no real c at all) or intersects \mathbb{R} at a single point.

Now, we fix an arbitrary integer c such that

$$\left| \frac{z_2 + c}{z_1 + c} \right|^2 \neq 1.$$

For any $\tau \in X$, the existence of the limit above implies that there is only a finite number of integers k such that $c_k(\tau) = c$. This means that any path $\tau \in X$ may contain only a finite number of boxes (p, q) on the diagonal $q - p = c$. Therefore, τ is contained in some subset of type $\mathcal{T}(p, q)$, which concludes the proof. \square

Lemma 8.4.2. *Let \tilde{A} and \tilde{B} be two central probability measures on \mathcal{T} and $(A^{(n)})$, $(B^{(n)})$ be the corresponding coherent systems. Assume $\tilde{A} \leq \text{const } \tilde{B}$ and let $f(\tau)$ denote the Radon–Nikodym derivative of \tilde{A} with respect to \tilde{B} . Assume further that $B^{(n)}(\lambda) \neq 0$ for all n and all $\lambda \in \mathbb{Y}_n$. Then*

$$\lim_{n \rightarrow \infty} \frac{A^{(n)}(\tau_n)}{B^{(n)}(\tau_n)} = f(\tau)$$

for almost all paths $\tau = (\tau_n) \in \mathcal{T}$ with respect to \tilde{B} .

Proof. Let $\mathcal{T}^{[n]}$ denote the set of finite paths in \mathbb{Y} going from \emptyset to a vertex in \mathbb{Y}_n . There is a natural projection $\mathcal{T} \rightarrow \mathcal{T}^{[n]}$ assigning to a path τ its finite part $\tau^{[n]} = (\tau_0, \dots, \tau_n)$. Notice that the infinite path space \mathcal{T} can be identified with the projective limit space $\varprojlim \mathcal{T}^{[n]}$.

Denote by $\Sigma^{[n]}$ the finite algebra of cylinder subsets with the bases in $\mathcal{T}^{[n]}$. The algebras $\Sigma^{[n]}$ form an increasing family, and the union $\Sigma = \bigcup \Sigma^{[n]}$ coincides with the algebra of all Borel sets with respect to the topology of \mathcal{T} .

Consider the probability space $(\mathcal{T}, \Sigma, \tilde{B})$. The function f is bounded and \tilde{B} -measurable. Hence, by the martingale theorem (cf., e.g., Shiryaev [Shir, Ch. VII, Section 4, Theorem 3])

$$\lim_{n \rightarrow \infty} \mathbb{E}(f \mid \Sigma^{[n]}) = f.$$

almost everywhere.

On the other hand, let $A^{[n]}$ and $B^{[n]}$ be the push-forwards of the measures \tilde{A} and \tilde{B} taken with respect to the projection $\mathcal{T} \rightarrow \mathcal{T}^{[n]}$. The conditional expectation $\mathbb{E}(f \mid \Sigma^{[n]})$ is nothing but the function

$$f_n(\tau) = \frac{A^{[n]}(\tau^{[n]})}{B^{[n]}(\tau^{[n]})}$$

Since \tilde{A} is a central measure, we have

$$A^{[n]}(\tau^{[n]}) = \frac{1}{\dim \tau_n} A^{(n)}(\tau_n),$$

and similarly

$$B^{[n]}(\tau^{[n]}) = \frac{1}{\dim \tau_n} B^{(n)}(\tau_n),$$

It follows

$$f_n(\tau) = \frac{A^{(n)}(\tau_n)}{B^{(n)}(\tau_n)},$$

and the proof is completed. \square

Proof of claim (i) of Theorem 8.1.2. We have to show that the measures \tilde{M}_1 and \tilde{M}_2 are disjoint. Set $\tilde{A} = \tilde{M}_1$, $\tilde{B} = (\tilde{M}_1 + \tilde{M}_2)/2$. Then $\tilde{A} \leq 2\tilde{B}$ and hence there exists the Radon–Nikodym derivative of \tilde{A} with respect to \tilde{B} . Denote it by $f(\tau)$. We have $0 \leq f(\tau) \leq 2$. The measures \tilde{M}_1 and \tilde{M}_2 are disjoint if and only if $f(\tau)$ takes only two values 0 and 1, almost surely with respect to the measure \tilde{B} .

On the other hand, by virtue of Lemma 8.4.2 above, $f(\tau)$ is \tilde{B} -almost surely the limit of the functions $f_n(\tau)$. Let Y be the set of those paths τ for which the limit of $f_n(\tau)$ exists and is distinct from 0 and 2. Observe that

$$\begin{aligned} f_n(\tau) &= \frac{A^{(n)}(\tau_n)}{B^{(n)}(\tau_n)} = 2 \frac{M_1^{(n)}(\tau_n)}{M_1^{(n)}(\tau_n) + M_2^{(n)}(\tau_n)} \\ &= 2 \left(1 + \frac{M_2^{(n)}(\tau_n)}{M_1^{(n)}(\tau_n)} \right)^{-1} = \frac{2}{1 + \varphi_n(\tau)}, \end{aligned}$$

in the notation introduced before Lemma 8.4.1. Consequently, Y coincides with the set of those paths τ for which $\varphi_n(\tau)$ has a finite nonzero limit, that is, $Y = X$. But $\tilde{M}_1(X) = \tilde{M}_2(X) = 0$ by virtue of Lemma 8.4.1. Hence, $\tilde{B}(Y) = B(X) = 0$, so that $f(\tau)$ is 0 or 2 almost surely with respect to \tilde{B} . \square

The proof of Theorem 8.1.2 is completed.

9. APPENDIX

9.1. Young diagrams and representations of finite symmetric groups. We identify partitions with Young diagrams, and denote by \mathbb{Y}_n the set of Young diagrams with n boxes. Given a Young diagram λ , we denote by $|\lambda|$ the number of its boxes, and by λ' the transposed diagram.

Recall that \mathbb{Y}_n is a natural set of labels for irreducible representations of the finite symmetric group $S(n)$. Given $\lambda \in \mathbb{Y}_n$, we denote by π^λ the corresponding irreducible representation of $S(n)$ and by χ^λ its character. Let

$$\dim \lambda = \dim \pi^\lambda = \chi^\lambda(e).$$

The quantity $\dim \lambda$ is called the *dimension* of λ . There are several explicit formulas for the dimension. For example, the *hook formula* says

$$\dim \lambda = \frac{n!}{\prod_{b \in \lambda} h(b)}, \quad \lambda \vdash n.$$

Here the symbol $b \in \lambda$ means that b is a box of λ , and if (i, j) are its coordinates than $h(b)$, the *hook length* of b , is defined by

$$h(b) = \lambda_i + \lambda'_j - i - j + 1.$$

Given two Young diagrams λ and μ , we write $\mu \nearrow \lambda$ if $\mu \subset \lambda$ and $|\lambda| = |\mu| + 1$, that is, λ is obtained from μ by adding a box.

The classical *Young branching rule* says that for any $\lambda \in \mathbb{Y}_n$

$$\pi^\lambda|_{S(n-1)} \sim \sum_{\mu \in \mathbb{Y}_{n-1} : \mu \nearrow \lambda} \pi^\mu.$$

This implies

$$\chi^\lambda|_{S(n-1)} = \sum_{\mu \in \mathbb{Y}_{n-1} : \mu \nearrow \lambda} \chi^\mu.$$

9.2. The Young graph and cotransition probabilities. Let \mathbb{Y} be the set of all Young diagrams: the disjoint union of the sets \mathbb{Y}_n , where $n = 0, 1, \dots$ (we agree that \mathbb{Y}_0 consists of a single element, the empty diagram \emptyset). We view \mathbb{Y} as the set of vertices of a graph, called the *Young graph* and denoted also by \mathbb{Y} . By definition, the edges of the Young graph are arbitrary couples $\mu \nearrow \lambda$. The Young graph is a convenient way to encode the Young branching rule.

For $\mu \in \mathbb{Y}_{n-1}$ and $\lambda \in \mathbb{Y}_n$ set

$$q(\mu, \lambda) = \begin{cases} \dim \mu / \dim \lambda, & \text{if } \mu \nearrow \lambda \\ 0, & \text{otherwise.} \end{cases}$$

By convention, $q(\emptyset, \lambda) = 1$ for the single element $\lambda \in \mathbb{Y}_1$ (the one-box diagram).

By the Young branching rule,

$$\sum_{\mu \in \mathbb{Y}_{n-1}: \mu \nearrow \lambda} \dim \mu = \dim \lambda,$$

so that

$$\sum_{\mu \in \mathbb{Y}_{n-1}, \lambda} q(\mu, \lambda) = 1.$$

The numbers $q(\mu, \lambda)$ are called the *cotransition probabilities* (see Kerov [Ker2], [Ker4]). They constitute a probability distribution for any fixed λ — the *cotransition distribution*.

9.3. Coherent systems of distributions on the Young graph. Let Δ_n be the set of probability distributions on the finite set \mathbb{Y}_n . This is a finite-dimensional simplex whose vertices are Dirac measures δ_λ with $\lambda \in \mathbb{Y}_n$.

For any $n = 1, 2, \dots$, define an affine map $\Delta_n \rightarrow \Delta_{n-1}$ by

$$\delta_\lambda \rightarrow \sum_{\mu \in \mathbb{Y}_{n-1}} q(\mu, \lambda) \delta_\mu.$$

Let

$$\Delta = \varprojlim \Delta_n$$

be the projective limit of the simplices taken with respect to these maps.

By the very definition, an element of Δ is a sequence $M = \{M^{(n)}\}_{n=0,1,\dots}$ such that $M^{(n)}$ is a probability distribution on \mathbb{Y}_n and any two measures $M^{(n-1)}, M^{(n)}$ with consecutive indices fulfill the *coherency relation*

$$M^{(n-1)}(\mu) = \sum_{\lambda \in \mathbb{Y}_n} q(\mu, \lambda) M^{(n)}(\lambda), \quad \forall \mu \in \mathbb{Y}_{n-1}.$$

Elements $M \in \Delta$ will be called *coherent systems of distributions*.

Alternatively, we may regard any $M \in \Delta$ as a real function on \mathbb{Y} such that

- First, $M(\lambda) \geq 0$ for any $\lambda \in \mathbb{Y}$.
- Second, for any $n = 1, 2, \dots$ and any $\mu \in \mathbb{Y}_{n-1}$

$$M(\mu) = \sum_{\lambda \in \mathbb{Y}_n} q(\mu, \lambda) M(\lambda).$$

- Third, $M(\emptyset) = 1$.

Indeed, the only point to be checked is that if M satisfies the above conditions then its restriction $M^{(n)} = M|_{\mathbb{Y}_n}$ is a probability distribution on \mathbb{Y}_n , that is, $\sum_{\lambda} M^{(n)}(\lambda) = 1$. But this is readily proved by induction on n .

A function $f(\lambda)$ on \mathbb{Y} is called *harmonic* if it satisfies the relation

$$f(\lambda) = \sum_{\nu \searrow \lambda} f(\nu) \quad \forall \lambda \in \mathbb{Y}.$$

The coherency relation for a function $M(\lambda)$ is equivalent to the harmonicity relation for the function $f(\lambda) = M(\lambda)/\dim \lambda$.

For further details, see Vershik–Kerov [VK2], Kerov [Ker2], [Ker4].

9.4. Central measures on paths and transition probabilities. By a *path* in the Young graph we mean a sequence of vertices

$$\tau = (\tau_n \nearrow \tau_{n+1} \nearrow \dots), \quad \tau_i \in \mathbb{Y}_i,$$

which may be finite or infinite. Let \mathcal{T} be the set of all infinite paths starting at \emptyset . This is a subset of the infinite product set $\prod_{n=0}^{\infty} \mathbb{Y}_n$. We endow \mathcal{T} with the induced topology. Then it becomes a compact totally disconnected topological space. Given a probability measure on the path space \mathcal{T} , we may speak about *random* paths.

To any coherent system M on \mathbb{Y} we assign a probability measure \widetilde{M} on \mathcal{T} with cylinder probabilities defined as follows. Let $\lambda \in \mathbb{Y}_n$ be an arbitrary vertex and τ be an arbitrary finite path going from \emptyset to λ , then the probability that the \widetilde{M} -random path goes along τ (up to λ) equals $M(\lambda)/\dim \lambda$.

The measure \widetilde{M} is *central* in the sense that the cylinder probabilities depend only on the final vertices λ but not on the paths τ_n chosen. Conversely, any central probability measure on the path space comes from a (unique) coherent system.

There is a useful characterizations of central measures as invariant measures with respect to a countable group of transformations of \mathcal{T} . This group is defined as follows. First, for each n we let $\mathcal{G}(n)$ be the group of the transformations $g : \mathcal{T} \rightarrow \mathcal{T}$ such that for any path $\tau = (\tau_n) \in \mathcal{T}$, we have $\tau_m = (g(\tau))_m$ for all $m \geq n$. Clearly, this is a finite group and we have $\mathcal{G}(n) \subset \mathcal{G}(n+1)$. Next, we define the group \mathcal{G} as the union of the groups $\mathcal{G}(n)$.

Proposition 9.4.1. *A measure on \mathcal{T} is central if and only if it is invariant under the action of \mathcal{G} .*

Define the *support* of a coherent system M as the subset

$$\text{supp}(M) = \{\lambda \in \mathbb{Y} : M(\lambda) \neq 0\} \subset \mathbb{Y}.$$

The measure \widetilde{M} is concentrated on the subspace of paths entirely contained in $\text{supp}(M)$. We may view \widetilde{M} as a Markov chain on the state set $\text{supp}(M)$, with the *transition probabilities*

$$p(\lambda, \nu) = \text{Prob}\{\tau_{n+1} = \nu \mid \tau_n = \lambda\}, \quad \lambda \in \mathbb{Y}_n, \quad \nu \in \mathbb{Y}_{n+1},$$

where $\tau = (\tau_n)$ is the random path. The transition probabilities $p(\lambda, \nu)$ are unambiguously defined for all $\lambda \in \text{supp}(M)$ by

$$p(\lambda, \nu) = \frac{M(\nu)}{M(\lambda)} \cdot \frac{\dim \lambda}{\dim \nu}, \quad \lambda \in \text{supp}(M).$$

The system of transition probabilities uniquely determines the initial central measure, so that distinct central measures have distinct transition probabilities. On the other hand, all central measures have one and the same system of cotransition probabilities, which are nothing but the quantities $q(\mu, \lambda)$ introduced in §9.2. That is, we have

$$q(\mu, \lambda) = \text{Prob}\{\tau_{n-1} = \mu \mid \tau_n = \lambda\}, \quad \mu \in \mathbb{Y}_{n-1}, \quad \lambda \in \mathbb{Y}_n.$$

For further details, see Vershik–Kerov [VK2], Kerov [Ker2], [Ker4].

9.5. Characters of the group $S(\infty)$. Recall that we have defined the *infinite symmetric group* $S(\infty)$ as the inductive limit of the finite symmetric group $S(n)$ as $n \rightarrow \infty$. A function $\chi : S(\infty) \rightarrow \mathbb{C}$ will be called a *character* if it is central (i.e. constant on conjugacy classes), positive definite, and normalized at the unity (i.e. $\chi(e) = 1$). The set of all characters of $S(\infty)$ will be denoted by \mathcal{X} .

Proposition 9.5.1. *There is a natural bijective correspondence*

$$\mathcal{X} \ni \chi \leftrightarrow M \in \Delta$$

between characters of $S(\infty)$ and coherent systems of probability measures on the Young graph.

Proof. Let $\chi \in \mathcal{X}$. For any n , set $\chi_n = \chi|_{S(n)}$. This is a central, positive definite, normalized function on $S(n)$. As readily verified, such functions are exactly the convex combinations of *normalized* irreducible characters $\chi^\lambda / \dim \lambda$. Thus,

$$\chi_n = \sum_{\lambda \in \mathbb{Y}_n} M^{(n)}(\lambda) \frac{\chi^\lambda}{\dim \lambda}$$

with certain nonnegative coefficients $M^{(n)}(\lambda)$,

$$\sum_{\lambda \in \mathbb{Y}_n} M^{(n)}(\lambda) = 1.$$

These coefficients may be viewed as the *Fourier coefficients* of the function χ_n . They form a probability distribution on \mathbb{Y}_n ; denote it by $M^{(n)}$. Let us check that the distributions $M^{(n)}$ obey the coherency relation. Indeed, by virtue of the Young branching rule, this is simply equivalent to saying that the function χ_{n-1} coincides with the restriction of χ_n to $S(n-1)$. Thus, starting from a character χ we obtain a coherent system $M = \{M^{(n)}\}$.

Conversely, let M be a coherent system. Then, for any n , we may define a function χ_n on $S(n)$ as above. These functions are pairwise compatible and hence define a function χ on the group $S(\infty)$. It is readily verified that χ is a character. \square

Both \mathcal{X} and Δ are certain sets of functions (on $S(\infty)$ and \mathbb{Y} , respectively). These sets are convex, and the bijection $\mathcal{X} \leftrightarrow \Delta$ is an isomorphism of convex sets.

Further, both \mathcal{X} and Δ are compact topological spaces with respect to the topology of pointwise convergence, and our bijection is a homeomorphism with respect to this topology.

9.6. Thoma's theorem. Let $\text{Ex } \mathcal{X}$ denote the set of extreme points of the convex set \mathcal{X} . Elements of $\text{Ex } \mathcal{X}$ will be called *extremal characters* of the group $S(\infty)$.

The first examples of extremal characters are as follows. Let $\Omega(p, q) \subset \mathbb{R}^p \times \mathbb{R}^q$ be the set of couples (α, β) , where $\alpha = (\alpha_1 \geq \dots \geq \alpha_p \geq 0)$ and $\beta = (\beta_1 \geq \dots \geq \beta_q \geq 0)$ be two collections of numbers such that

$$\sum_{i=1}^p \alpha_i + \sum_{j=1}^q \beta_j = 1.$$

Here one of the numbers p, q may be zero (then the corresponding collection α or β disappears).

For $(\alpha, \beta) \in \Omega(p, q)$ and $k = 1, 2, \dots$, set

$$\tilde{p}_k(\alpha, \beta) = \sum_{i=1}^p \alpha_i^k + (-1)^{k-1} \sum_{j=1}^q \beta_j^k.$$

Notice that

$$\tilde{p}_1(\alpha, \beta) \equiv 1.$$

Given $s \in S(\infty)$, we denote by $m_k(s)$ the number of k -cycles in s . Since s is a finite permutation, we have

$$m_1(s) = \infty, \quad m_k(s) < \infty \text{ for } k \geq 2, \quad m_k(s) = 0 \text{ for } k \text{ large enough.}$$

In this notation, we define a function on $S(\infty)$ by

$$\chi^{(\alpha, \beta)}(s) = \prod_{k=1}^{\infty} (\tilde{p}_k(\alpha, \beta))^{m_k(s)} = \prod_{k=2}^{\infty} (\tilde{p}_k(\alpha, \beta))^{m_k(s)}, \quad s \in S(\infty),$$

where we agree that $1^\infty = 1$ and $0^0 = 1$. Any such function turns out to be an extremal character: this claim is a particular case of a more general result stated below.

If $p = 1$ and $q = 0$ (i.e., $\alpha_1 = 1$ and all other parameters disappear) then we get the trivial character, which equals 1 identically. If $p = 0$ and $q = 1$ then we get the alternate character $\text{sgn}(s) = \pm 1$, where the plus-minus sign is chosen according to the parity of the permutation. More generally, we have

$$\chi^{(\alpha, \beta)} \cdot \text{sgn} = \chi^{(\beta, \alpha)}.$$

Let \mathbb{R}^∞ denote the direct product of countably many copies of \mathbb{R} . We equip \mathbb{R}^∞ with the product topology. Let Ω be the subset of $\mathbb{R}^\infty \times \mathbb{R}^\infty$ formed by couples $\alpha \in \mathbb{R}^\infty, \beta \in \mathbb{R}^\infty$ such that

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq 0), \quad \beta = (\beta_1 \geq \beta_2 \geq \dots \geq 0), \quad \sum_{i=1}^{\infty} \alpha_i + \sum_{j=1}^{\infty} \beta_j \leq 1.$$

We call Ω the *Thoma simplex*. As affine coordinates of the simplex one can take the numbers

$$\alpha_1 - \alpha_2, \dots, \alpha_{p-1} - \alpha_p, \alpha_p, \beta_1 - \beta_2, \dots, \beta_{q-1} - \beta_q, \beta_q$$

but we will not use these coordinates. We equip Ω with topology induced from that of the space $\mathbb{R}^\infty \times \mathbb{R}^\infty$. It is readily seen that Ω is a compact space. Clearly, each set $\Omega(p, q)$ may be viewed as a subset of Ω (this is one of finite-dimensional faces of Ω).

Notice that the union of the simplices $\Omega(p, q)$ is dense in Ω . For instance, the point $(\underline{0}, \underline{0}) = (\alpha \equiv 0, \beta \equiv 0) \in \Omega$ can be approximated by points of the simplices Ω_{p0} as $p \rightarrow \infty$,

$$(\underline{0}, \underline{0}) = \lim_{p \rightarrow \infty} (\underbrace{(1/p, \dots, 1/p)}_p, \underline{0}).$$

Now we extend by continuity the definition of the functions $\chi^{(\alpha, \beta)}$ given above. First, for any $k = 2, 3, \dots$ we define the function \tilde{p}_k on Ω as follows. If $\omega = (\alpha, \beta) \in \Omega$ then

$$\tilde{p}_k(\omega) = \tilde{p}_k(\alpha, \beta) = \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \sum_{j=1}^{\infty} \beta_j^k.$$

Note that \tilde{p}_k is a continuous function on Ω . It should be emphasized that the condition $k \geq 2$ is necessary here: the similar expression with $k = 1$ (that is, the sum of all coordinates) *is not* continuous.

Next, for any $\omega = (\alpha, \beta) \in \Omega$ we set

$$\chi^{(\omega)}(s) = \chi^{(\alpha, \beta)}(s) = \prod_{k=2}^{\infty} (\tilde{p}_k(\alpha, \beta))^{m_k(s)}, \quad s \in S(\infty),$$

Theorem 9.6.1 (Thoma's theorem). *The functions $\chi^{(\omega)}$ are precisely the extremal characters of the group $S(\infty)$.*

That is, for any $\omega \in \Omega$ the function $\chi^{(\omega)}$ is an extremal character, each extremal character is obtained in this way, and different points $\omega \in \Omega$ define different characters.

In particular, the character $\chi^{(\underline{0}, \underline{0})}$ is the delta function at $e \in S(\infty)$. It corresponds to the biregular representation defined in §1.3.

Notice that the set \mathcal{X} carries a natural topology – that of pointwise convergence on the group. Endow the subset $\text{Ex } \mathcal{X} \subset \mathcal{X}$ with the induced topology. Then the correspondence $\text{Ex } \mathcal{X} \leftrightarrow \Omega$ given by Thoma's theorem becomes a homeomorphism of topological spaces.

This implies, in particular, that characters $\chi^{(\omega)}$ with parameters $\omega \in \cup_{p, q} \Omega(p, q)$ are dense in the whole set $\text{Ex } \mathcal{X}$ with respect to the topology of pointwise convergence on the group $S(\infty)$.

Comments to Thoma's theorem. 1. The original proof of Thoma was given in his paper [Tho1] published in 1964. Thoma first proved (Satz 1 in [Tho1]) that a character χ is extremal if and only if it is a multiplicative class function, that is,

$$\chi(s) = \prod_{k=2}^{\infty} p_k^{m_k(s)}, \quad s \in S(\infty),$$

with certain real numbers p_2, p_3, \dots . This reduced the problem to the following one: find all sequences (p_2, p_3, \dots) such that the expression above is a positive

definite function on the group $S(\infty)$. An equivalent condition on (p_2, p_3, \dots) is as follows: let h_1, h_2, \dots be defined by

$$1 + h_1 u + h_2 u^2 + h_3 u^3 + \dots = \exp \left(u + \sum_{k=2}^{\infty} p_k \frac{u^k}{k} \right)$$

and set $h_0 = 1, h_{-1} = h_{-2} = \dots = 0$; then

$$\det_{1 \leq i, j \leq \ell(\lambda)} [h_{\lambda_i - i + j}] \geq 0 \quad \text{for any } \lambda \in \mathbb{Y}.$$

Then Thoma succeeded to prove that the sequences $(h_1 = 1, h_2, h_3, \dots)$ with this property are exactly those given by the formula

$$1 + h_1 u + h_2 u^2 + h_3 u^3 + \dots = e^{\gamma u} \prod_{i=1}^{\infty} \frac{1 + \beta_i u}{1 - \alpha_i u},$$

where

$$(\alpha, \beta) \in \Omega, \quad \gamma = 1 - \sum_{i=1}^{\infty} (\alpha_i + \beta_i),$$

which implies the theorem. Actually, this result is equivalent to Edrei's classification (1952) of one-sided, totally positive sequences in the sense of Schoenberg, see Edrei [Edr]. Since the paper [Tho1] contains no reference to Schoenberg or Edrei, one may conclude that Thoma was unaware of their work.

2. Quite a different proof of Thoma's theorem was given by Vershik and Kerov ([VK1], [VK2], 1981). Instead of function-theoretic arguments of Edrei and Thoma, Vershik and Kerov used an asymptotic method (whose general idea was suggested by Vershik's paper [Ver]): approximation of extremal characters of $S(\infty)$ by irreducible normalized characters $\chi^\lambda / \dim \lambda$ of finite groups $S(n)$, as $n \rightarrow \infty$. The asymptotic method explains the origin of Thoma's parameters α_i, β_i : they arise as limits of normalized Frobenius coordinates of the growing diagram λ .

3. An important combinatorial lemma, stated in [VK2] without proof (see [VK2, §5, Lemma 1]), was proved in Kerov–Olshanski [KO]. A particular case of it (which is sufficient for completing the proof of Thoma's theorem) was proved in Wassermann's thesis ([Was]). For more detail, see also Okounkov–Olshanski [OkOl, §8], Olshanski–Regev–Vershik [ORV].

4. Okounkov's work [Ok1, Ok2] provides one more approach to Thoma's theorem. In particular, Okounkov showed that a crucial step in Thoma's proof can be replaced by a simple representation-theoretic argument.

5. Finally, the paper Kerov–Okounkov–Olshanski [KOO] contains a far generalization of Thoma's theorem obtained by the asymptotic method of [VK2].

9.7. Spectral decomposition of characters.

Theorem 9.7.1. (i) *For any character $\chi \in \mathcal{X}$, there exists a probability measure σ on the Thoma simplex Ω such that*

$$\chi(s) = \int_{\Omega} \chi^{(\omega)}(s) \sigma(d\omega), \quad s \in S(\infty).$$

(ii) *Such a measure is unique.*

(iii) *Conversely, for any probability measure σ on Ω , the function χ defined by the above formula is a character of $S(\infty)$.*

Thus, \mathcal{X} is isomorphic, as a convex set, to the set of all probability measures on the compact space Ω .

We call this integral representation the *spectral decomposition* of a character. The measure σ will be called the *spectral measure* of χ . If χ is extremal, i.e., $\chi = \chi^{(\omega)}$, then its spectral measure reduces to the Dirac mass at ω .

Theorem 9.7.1 admits an equivalent formulation in terms of coherent systems. To state it we need to extend the above definition of the functions $\tilde{p}_k(\omega)$ to arbitrary symmetric functions. Let Λ be the algebra of symmetric functions (see Macdonald [Mac]). The power-sums p_k are algebraically independent generators of Λ , so that the assignment $p_k \mapsto \tilde{p}_k$ can be extended to a homomorphism of the algebra Λ into the algebra $C(\Omega)$ of continuous functions on the Thoma simplex Ω . Given $f \in \Lambda$, we denote by \tilde{f} its image in $C(\Omega)$. In particular, we apply this to the Schur functions s_λ : the corresponding functions $\tilde{s}_\lambda(\omega)$ are called the *extended Schur functions* (see Vershik–Kerov [VK3, §6], Kerov–Okounkov–Olshanski [KOO, Appendix]). Notice that the restriction of \tilde{s}_λ to $\Omega(p, q) \subset \Omega$ is nothing but the *supersymmetric* Schur polynomial $s_\lambda(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q)$ (see, e.g., Berele–Regev [BR], Macdonald [Mac, Example I.3.23]; in Macdonald’s notation, this is $s_\lambda(\alpha_1, \dots, \alpha_p; -\beta_1, \dots, -\beta_q)$).

Theorem 9.7.2. *Let χ be a character of $S(\infty)$, $M = (M^{(n)})$ be the corresponding coherent system, and σ be the spectral measure of χ . For any $n = 0, 1, 2, \dots$ and any $\lambda \in \mathbb{Y}_n$ we have*

$$M^{(n)}(\lambda) = \int_{\Omega} \dim \lambda \cdot \tilde{s}_\lambda(\omega) \sigma(d\omega).$$

In particular, the coherent system corresponding to an extremal character $\chi^{(\omega)}$ has the form $M^{(n)}(\lambda) = \dim \lambda \cdot \tilde{s}_\lambda(\omega)$.

For a proof of Theorem 9.7.2, see Kerov–Okounkov–Olshanski [KOO] (that paper actually contained a more general result).

The claim of the theorem is similar to the Poisson integral representation of harmonic functions (see Kerov [Ker2], [Ker4], Kerov–Okounkov–Olshanski [KOO]). Notice that the role of the Poisson kernel is played here by the function $(\lambda, \omega) \mapsto \tilde{s}_\lambda(\omega)$.

Theorems 9.7.1 and 9.7.2 involve the claim that a certain convex set (that of characters or coherent systems, or yet equivalently, that of central measures) is a *Choquet simplex*. That is, each point of the convex set in question is *uniquely* representable by a probability measure on the subset of extreme points. This fact does not rely on the specific nature of the group $S(\infty)$ or the Young graph, and can be derived from some very general theorems. See, e.g., Diaconis–Freedman [DS, section 4], Olshanski [Ol7, §9].

The next result can be viewed as a kind of Fatou’s theorem on boundary values of harmonic functions. To state it we need to introduce some important notation and definitions.

Recall the definition of the *Frobenius coordinates* of a nonempty diagram $\lambda \in \mathbb{Y}$: these are the integers $p_1 > \cdots > p_d \geq 0$, $q_1 > \cdots > q_d \geq 0$, where d is the number of boxes on the main diagonal of λ and

$$p_i = \lambda_i - i, \quad q_i = \lambda'_i - i, \quad i = 1, \dots, d.$$

Any collection of integers $p_1 > \cdots > p_d \geq 0$, $q_1 > \cdots > q_d \geq 0$ corresponds to a Young diagram. The transposition $\lambda \mapsto \lambda'$ corresponds to interchanging $p_i \leftrightarrow q_i$.

We also need the so called *modified* Frobenius coordinates of a diagram λ , which are defined by

$$\tilde{p}_i = p_i + \frac{1}{2}, \quad \tilde{q}_i = q_i + \frac{1}{2}, \quad i = 1, \dots, d.$$

It is convenient to agree that

$$\tilde{p}_i = \tilde{q}_i = 0, \quad i > d.$$

Notice that

$$\sum_{i=1}^{\infty} (\tilde{p}_i + \tilde{q}_i) = |\lambda|.$$

For each n we define an embedding $\iota_n : \mathbb{Y}_n \rightarrow \Omega$ by

$$\iota_n(\lambda) = (\alpha, \beta), \quad \alpha_i = \frac{\tilde{p}_i}{n}, \quad \beta_i = \frac{\tilde{q}_i}{n}, \quad i = 1, 2, \dots, \quad \lambda \in \mathbb{Y}_n.$$

Notice that the union of the finite sets $\iota_n(\mathbb{Y}_n)$ is dense in Ω .

Theorem 9.7.3. *Let χ be a character of $S(\infty)$, σ be its spectral measure, and $M = (M^{(n)})$ be the coherent system of distributions corresponding to χ . Further, let $\iota_n(M^{(n)})$ be the push-forward of the measure $M^{(n)}$ under the embedding $\iota_n : \mathbb{Y}_n \rightarrow \Omega$.*

As $n \rightarrow \infty$, the measures $\iota_n(M^{(n)})$ converge to σ in the weak topology of measures on the compact space Ω .

For a proof, see Kerov–Okounkov–Olshanski [KOO].

Finally, we shall state a related result concerning central measures. An infinite path $\tau = (\tau_n) \in \mathcal{T}$ is called *regular* if the points $\iota_n(\tau_n) \in \Omega$ converge to a limit as $n \rightarrow \infty$; then the limit point is called the *end* of the path. Let \mathcal{T}' be the set of regular paths; this is a Borel subset of \mathcal{T} . Assigning to a regular path its end we get a Borel map $\mathcal{T}' \rightarrow \Omega$.

Theorem 9.7.4. *Any central probability measure \tilde{M} is supported by \mathcal{T}' . The push-forward of \tilde{M} under the map $\mathcal{T}' \rightarrow \Omega$ coincides with the spectral measure σ of the character $\chi \leftrightarrow \tilde{M}$.*

The proof is similar to (and actually simpler than) the proof of Theorem 10.2 in Olshanski [Olsh7].

9.8. Spherical representations and spherical functions. Let G be the bisymmetric group $S(\infty) \times S(\infty)$ and K be the diagonal subgroup in G , canonically isomorphic to $S(\infty)$.

Assume we are given a unitary representation T of the group G in a Hilbert space \mathcal{H} . A vector $\xi \in \mathcal{H}$ is said to be a *cyclic vector* if the linear span of the vectors

$T(g)\xi$, where g ranges over G , is dense in \mathcal{H} . Suppose that ξ is cyclic, invariant under the action of the subgroup K , and $\|\xi\| = 1$. In such a case we say that the couple (T, ξ) is a *spherical representation*. We will call ξ the *spherical vector*.

Denote by Φ the set of all functions on G that are positive definite, K -bi-invariant, and normalized at the unity. If (T, ξ) is a spherical representation of (G, K) , then the matrix coefficient corresponding to the spherical vector,

$$\varphi(g) = (T(g)\xi, \xi), \quad g \in G,$$

is an element of Φ . We call φ the *spherical function* of (T, ξ) . The couple (T, ξ) is uniquely (up to a natural equivalence) reconstructed from its spherical function, by use of the Gelfand–Naimark construction. Moreover, any $\varphi \in \Phi$ comes from a certain (T, ξ) . Thus, there is a one-to-one correspondence between functions $\varphi \in \Phi$ and (equivalence classes of) spherical representations (T, ξ) .

Assume T is an irreducible unitary representation of G . Then the space of its K -invariant vectors has dimension 0 or 1 (indeed, this follows from the fact that (G, K) is a Gelfand pair, see Olshanski [Ol3, §1]). Thus, if T possesses a nonzero K -invariant vector ξ then ξ is unique, within a scalar multiple. Observe that ξ is automatically cyclic, because any nonzero vector in an irreducible representation is cyclic. Thus, assuming $\|\xi\| = 1$, we see that (T, ξ) is a spherical representation. The only lack of uniqueness in the choice of ξ is reduced to multiplying ξ by a complex scalar of absolute value 1, which does not affect the spherical function $\varphi(g) = (T(g)\xi, \xi)$. Notice that φ is an extreme point of the convex set Φ . Conversely, if $\varphi \in \Phi$ is extreme then the corresponding spherical representation is irreducible.

Proposition 9.8.1. *There is a natural bijective correspondence $\chi \leftrightarrow \varphi$ between characters $\chi \in \mathcal{X}$ and spherical functions $\varphi \in \Phi$.*

Proof. Indeed, the relation between χ and φ has the form

$$\varphi(g_1, g_2) = \chi(g_1 g_2^{-1}), \quad \chi(s) = \varphi(s, e),$$

where g_1, g_2, s are elements of $S(\infty)$. Clearly, the normalization $\chi(e) = 1$ is equivalent to the normalization $\varphi(e) = 1$. It is readily verified that χ is constant on conjugacy classes if and only if φ is constant on double cosets. Next, let $\{g_i\} = \{(g_{i1}, g_{i2})\}$ be a finite collection of element of the group G , and let $s_i = g_{i1} g_{i2}^{-1}$ be the corresponding elements in $S(\infty)$. Remark that $g_j^{-1} g_i$ lies in the same double coset modulo K as $(s_j^{-1} s_i, e)$. It follows that $\varphi(g_j^{-1} g_i) = \chi(s_j^{-1} s_i)$, so that φ is positive definite if and only if χ is. Thus, $\chi \leftrightarrow \varphi$ is indeed a bijective correspondence between the two sets. \square

Clearly, the bijection $\chi \leftrightarrow \varphi$ is an isomorphism of convex set. Therefore, irreducible spherical representations of (G, K) are parametrized by extremal characters, and finally by points $\omega \in \Omega$.

More generally, combining Proposition 9.8.1 with the description of characters given in Theorem 9.7.1 we obtain a general description of spherical representations (T, ξ) . Specifically, any such (T, ξ) is determined by a probability measure σ on Ω . It is worth noting that, as long we are dealing with reducible spherical representations, a given T may well possess a lot of K -invariant cyclic vectors. If ξ is replaced by another spherical vector ξ' then σ is replaced by an equivalent probability measure

σ' . Thus, the equivalence class of T is determined by the equivalence class of σ . This equivalence class of measures on Ω will be called the *spectral type of T* .

Using the abstract machinery of direct integrals of Hilbert spaces one can show that any spherical representation (T, ξ) can be decomposed into a multiplicity free direct integral of irreducible spherical representations $(T^{(\omega)}, \xi^\omega)$. This decomposition, which may be called the *spectral decomposition of (T, ξ)* , is unique, and it is governed by the spectral measure σ :

$$T = \int_{\omega} T^{(\omega)} \sigma(d\omega), \quad \xi = \int_{\omega} \xi^{(\omega)} \sigma(d\omega).$$

Assume (T', ξ') is another spherical representation and σ' is the corresponding spectral measure. The measures σ and σ' are said to be *disjoint* if they are singular with respect to each other (then there exist two disjoint Borel sets supporting them). The representations T and T' are said to be *disjoint* if they have no equivalent nonzero subrepresentations. Disjointness of σ and σ' is equivalent to disjointness of T and T' .

9.9. Admissible representations. Spherical representations enter a wider class of representations that will be defined now.

For $m \leq n$, let $S_m(n) \subseteq S(n)$ denote the subgroup fixing the points $1, 2, \dots, m$. Set

$$S_m(\infty) = \bigcup_{n \geq m} S_m(n) \subset S(\infty)$$

and denote by $K_m \subset K$ the corresponding subgroup of $K \cong S(\infty)$. Recall that by $G(m)$ we denote the subgroup $S(m) \times S(m)$ in the bi-symmetric group G . An important fact is that the subgroups K_m and $G(m)$ commute to each other.

Given a unitary representation T of the group G in a Hilbert space \mathcal{H} , let $\mathcal{H}_m = \mathcal{H}^{K_m}$ be the subspace of K_m -invariant vectors, and set

$$\mathcal{H}_\infty = \bigcup_m \mathcal{H}_m.$$

We remark that \mathcal{H}_∞ is a G -invariant (algebraic) subspace in \mathcal{H} . Indeed, for any m , \mathcal{H}_m is invariant under $G(m)$, because $G(m)$ and K_m commute. Since G is the union of $G(m)$'s, it follows that \mathcal{H}_∞ is invariant under G . Thus, the closure of \mathcal{H}_∞ is an invariant subspace of the representation T .

We say that T is an *admissible* representation of the pair (G, K) if the subspace \mathcal{H}_∞ as defined above is dense in \mathcal{H} .

For more detail about this definition, see [Ol3] and also [Ol2], [Ol4]. Not all representations of G are admissible, for it may well happen that \mathcal{H}_∞ is reduced to $\{0\}$. If T is irreducible then either it is admissible or $\mathcal{H}_\infty = \{0\}$.

Any spherical representation (T, ξ) is admissible. Indeed, the spherical vector ξ belongs to the subspace \mathcal{H}_∞ . Therefore, all vectors $T(g)\xi$, where $g \in G$, are also in \mathcal{H}_∞ . Since these vectors generate a dense algebraic subspace, \mathcal{H}_∞ is dense in \mathcal{H} , so that T is admissible.

As shown in [Ol3], admissible representations are exactly those unitary representations that can be continuously extended to the topological group $\overline{G} \supset G$. Thus, admissible representations of (G, K) are in essence the same as continuous

unitary representations of the group \overline{G} . However, for technical reasons, admissible representations are more convenient to deal with than representations of \overline{G} .

Any admissible representation is a type I representation, i.e., the von Neumann algebra generated by it is of type I ([Ol3, Theorem 4.1]). This means that inside the class of admissible representation there is no pathologies occurring for general representations of non-tame groups. Irreducible admissible representations admit a complete classification (Okounkov [Ok1], [Ok2]), their explicit realization is described in [Ol3, §5].

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